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Vladimir Turaev

Homotopy Quantum Field Theory

With Appendices by
Michael Müger and Alexis Virelizier

European Mathematical Society

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Dedicated to Mitja, Marie-Catherine, and Marc

Contents

Introduction

Topological quantum field theories (TQFTs) produce topological invariants of manifolds using ideas suggested by quantum field theory; see [At], [Wi1]. For $d \geq 0$, a $(d+1)$-dimensional TQFT over a commutative ring K assigns to every closed oriented d-dimensional manifold M a projective K-module of finite type A_M and assigns to every compact oriented $(d+1)$-dimensional cobordism (W, M_0, M_1) a K-homomorphism $\tau(W)\colon A_{M_0} \to A_{M_1}$. These modules and homomorphisms should satisfy several axioms including tensor multiplicativity with respect to disjoint union and functoriality with respect to gluing of cobordisms.

The study of TQFTs has been especially successful in low dimensions $d = 0, 1, 2, 3$. One-dimensional TQFTs ($d = 0$) bijectively correspond to projective K-modules of finite type. Two-dimensional TQFTs ($d = 1$) are fully classified in terms of commutative Frobenius algebras, see [Di], [Ab], [Kock]. Three-dimensional TQFTs ($d = 2$) are closely related to quantum groups and braided categories; see [RT], [Tu2], [KRT], [BK]. Powerful four-dimensional TQFTs ($d = 3$) arise from the Heegaard–Floer homology of 3-manifolds due to P. Ozsváth and Z. Szabó; see [OS1], [OS2]. Algebraic structures underlying four-dimensional TQFTs are yet to be unraveled; for work in this direction see [CKY], [CKS], [CJKLS], [Ma], [Oe].

In this monograph we apply the idea of a TQFT to maps from manifolds to topological spaces. This leads us to a notion of a $(d+1)$-dimensional homotopy quantum field theory (HQFT) which may be described as a TQFT for closed oriented d-dimensional manifolds and compact oriented $(d+1)$-dimensional cobordisms endowed with maps to a given space X. Such an HQFT yields numerical homotopy invariants of maps from closed oriented $(d+1)$-dimensional manifolds to X. A TQFT may be interpreted in this language as an HQFT with target space consisting of one point. The general notion of a $(d+1)$-dimensional HQFT was introduced in 1999 in my unpublished preprint [Tu3] and independently by M. Brightwell and P. Turner [BT1] for $d = 1$ and simply connected target spaces.

If the ground ring K is a field, then the $(0+1)$-dimensional HQFTs with target X correspond bijectively to finite-dimensional representations of the fundamental group of X or, equivalently, to finite-dimensional flat K-vector bundles over X. This allows one to view HQFTs as high-dimensional generalizations of flat vector bundles.

We shall mainly study the case where $X = K(G, 1)$ is the Eilenberg–MacLane space corresponding to a (discrete) group G. A manifold endowed with a homotopy class of maps to such X will be called a G-manifold. The maps to $X = K(G, 1)$ classify principal G-bundles, and numerical invariants of principal G-bundles over closed oriented $(d+1)$-dimensional manifolds provided by HQFTs with target X can be viewed as "quantum" characteristic numbers. From this perspective, the standard Witten–Reshetikhin–Turaev quantum invariants of 3-manifolds can be regarded as quantum characteristic numbers of the trivial bundles over 3-manifolds.

The main goal of this monograph is a construction of $(d+1)$-dimensional HQFTs with target $K(G,1)$ for $d = 1, 2$. We focus on algebraic structures underlying such HQFTs. For $d = 1$, these structures are formulated in terms of G-graded algebras or, briefly, G-algebras. A G-algebra is an associative unital algebra L endowed with a decomposition $L = \bigoplus_{\alpha\in G} L_\alpha$ such that $L_\alpha L_\beta \subset L_{\alpha\beta}$ for any $\alpha, \beta \in G$. The G-algebras arising from 2-dimensional HQFTs have additional features including a natural inner product and an action of G. This leads us to a notion of a crossed Frobenius G-algebra. Our main result concerning 2-dimensional HQFTs with target $K(G,1)$ is a bijective correspondence between the isomorphism classes of such HQFTs and the isomorphism classes of crossed Frobenius G-algebras. This generalizes the standard equivalence between 2-dimensional TQFTs and commutative Frobenius algebras (the case $G = 1$). Our second result is a classification of semisimple crossed Frobenius G-algebras in terms of 2-dimensional cohomology classes of the subgroups of G of finite index.

The study of 2-dimensional HQFTs has not yet brought to light new invariants of principal G-bundles over surfaces. The invariants arising from semisimple (crossed Frobenius) G-algebras are essentially homological. The invariants arising from non-semisimple G-algebras may in principle be new but are poorly understood. On the other hand, the study of 2-dimensional HQFTs finds interesting applications in certain enumeration problems. One of these problems concerns an arbitrary Serre fibration $p\colon E \to W$ over a closed connected oriented surface W of positive genus. The bundle p may have sections, i.e., continuous mappings $s\colon W \to E$ such that $ps = \mathrm{id}_W$ (we work in the pointed category so that E and W have base points preserved by p and s). Assume that the fiber F of p is path-connected and the group $\pi_1(F)$ is finite. Then the number of sections of p, considered up to homotopy and a natural action of $\pi_2(F)$, is finite (possibly zero). We give a formula expressing this number in terms of 2-dimensional cohomology classes associated with irreducible complex linear representations of $\pi_1(F)$. The definition of these cohomology classes and the statement of our formula are direct and do not involve HQFTs, while the proof heavily uses HQFTs. This yields, in particular, the following solution to the existence problem for sections: p has a section if and only if the number produced by our formula is non-zero. As a specific application, note the following theorem: in the case where the group $\pi_1(F)$ is abelian (and finite), the bundle $p\colon E \to W$ has a section if and only if the induced homomorphism $p_*\colon H_2(E;\mathbb{Z}) \to H_2(W;\mathbb{Z})$ is surjective. A similar theorem holds for any finite group $\pi_1(F)$ whose order is small with respect to the genus of W. Other topological applications concern principal bundles over W and non-abelian 1-cohomology of W. For group-theoretic applications (not discussed in the book), the reader is referred to [NT].

More generally, given a cohomology class $\theta \in H^2(E;\mathbb{C}^*)$, we can provide each section s of p with the weight $\theta(s_*([W])) = s^*(\theta)([W]) \in \mathbb{C}^* = \mathbb{C} - \{0\}$, where $[W] \in H_2(W;\mathbb{Z})$ is the fundamental class of W. Counting sections of p with these weights we obtain a complex number viewed as a θ-weighted number of sections of p. We express this number in terms of 2-dimensional cohomology classes associated with irreducible complex projective representations of $\pi_1(F)$.

The enumeration problem for sections formulated above is equivalent to a special case of the following problem. Consider a group epimorphism $G' \to G$ with finite kernel Γ. Consider a homomorphism g from the fundamental group of a closed connected oriented surface W to G. It is clear that g has only a finite number of lifts to G' (if any). How to compute this number? For the trivial homomorphism $g = 1$, a solution is given by the Frobenius–Mednykh formula; see [Fr], [Me]. We extend this formula to arbitrary g. Our formula computes the number of lifts of g in terms of 2-dimensional cohomology classes associated with irreducible complex linear representations of Γ. A more general formula counts the lifts of g with weights determined by an element of $H^2(G'; \mathbb{C}^*)$.

Our approach to 3-dimensional HQFTs is based on a connection between braided categories and knots. This connection is essential in the construction of topological invariants of knots and 3-manifolds from quantum groups. We extend this train of ideas to links $\ell \subset S^3$ endowed with homomorphisms $\pi_1(S^3 \backslash \ell) \to G$ and to 3-dimensional G-manifolds. To this end, we introduce crossed G-categories and study braidings and twists in such categories. This leads us to a notion of a modular crossed G-category.

We show that each modular crossed G-category gives rise to a three-dimensional HQFT with target $K(G, 1)$. This HQFT has two ingredients: a "homotopy modular functor" A assigning projective K-modules to G-surfaces and a functor τ assigning K-homomorphisms to 3-dimensional G-cobordisms. In particular, the HQFT provides numerical invariants of closed oriented 3-dimensional G-manifolds. For $G = 1$, we recover the familiar construction of 3-dimensional TQFTs from modular categories; see [Tu2].

We discuss several algebraic methods producing crossed G-categories. In particular, we introduce quasitriangular Hopf G-coalgebras and show that they give rise to crossed G-categories. However, the problem of systematic finding of modular crossed G-categories is largely open.

This book is based on my papers [Tu3]–[Tu8]. Chapters I–IV cover [Tu3], though the exposition has been somewhat modified and Sections IV.1, IV.2 added. Chapter V covers [Tu5]–[Tu7]. Chapters VI, VII, and VIII cover [Tu4] and [Tu8]. These three chapters extend the first part of my monograph [Tu2]. Though techniques from [Tu2] are used in several proofs in Chapters VI and VII, the definitions and statements of theorems can be understood without knowledge of [Tu2]. The reader's background is supposed only to include basics of algebra and topology and (starting from Chapter VI) basics of the theory of categories.

Here is a chapter-wise description of the book. In Chapter I we discuss a general setting of HQFTs. In particular, we show that $(d+1)$-dimensional cohomology classes of the target space and of its finite-sheeted coverings give rise to $(d+1)$-dimensional HQFTs. These HQFTs and their direct sums are called cohomological HQFTs. For $d = 1$, we introduce a wider class of semi-cohomological HQFTs.

In Chapter II we introduce and study G-algebras. We discuss various classes of G-algebras including Frobenius, crossed, semisimple, Hermitian, and unitary G-algebras. The main result of this chapter is a classification of semisimple crossed Frobenius G-algebras over a field of characteristic zero.

In Chapter III we associate with each 2-dimensional HQFT with target $K(G,1)$ an underlying crossed Frobenius G-algebra. This establishes a bijection between the isomorphism classes of 2-dimensional HQFTs with target $K(G,1)$ and the isomorphism classes of crossed Frobenius G-algebras. Under this bijection, the semi-cohomological HQFTs correspond to semisimple algebras at least when the ground ring is a field of characteristic zero. We also establish a Verlinde-type formula for the semi-cohomological HQFTs.

In Chapter IV we introduce biangular G-algebras. They give rise to lattice models for 2-dimensional HQFTs generalizing the lattice models for 2-dimensional TQFTs introduced in [BP], [FHK]. We prove that the lattice 2-dimensional HQFTs over algebraically closed fields of characteristic zero are semi-cohomological.

In Chapter V we discuss applications of HQFTs to enumeration problems. This chapter is almost entirely independent of the previous chapters except the proof of the main theorem given in Sections V.5 and V.6.

In Chapter VI we introduce crossed G-categories and various additional structures on them (braiding, twist, etc.). We also introduce G-links, G-tangles, and G-graphs in $\mathbb{R}^3$ and define their colorings by objects and morphisms of a ribbon crossed G-category $\mathcal{C}$. The $\mathcal{C}$-colored G-graphs form a monoidal tensor category, and we define a canonical monoidal functor from this category to $\mathcal{C}$. The chapter ends with a study of dimensions of objects and traces of morphisms in $\mathcal{C}$.

In Chapter VII we introduce modular crossed G-categories. Each such category produces a 3-dimensional HQFT with target $K(G,1)$.

In Chapter VIII we introduce Hopf G-algebras and discuss algebraic constructions of crossed G-categories and crossed G-algebras.

The book ends with seven Appendices. Appendix 1 is concerned with relative HQFTs generalizing the so-called open-closed TQFTs. In Appendix 2 we outline a state sum approach to invariants of 3-dimensional G-manifolds. In Appendix 3 we briefly discuss recent developments in the study of HQFTs and related areas. In Appendix 4 we formulate several open problems. Appendix 5 written by Michael Müger is concerned with his recent work on braided crossed G-categories. Appendices 6 and 7 written by Alexis Virelizier discuss algebraic properties of Hopf G-coalgebras and the 3-manifolds invariants derived from Hopf G-coalgebras.

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Throughout the book, the symbol K denotes a commutative ring with unit 1_K. The multiplicative group of invertible elements of K is denoted K^*. The symbol G denotes a (discrete) group.

Chapter I
Generalities on HQFTs

I.1 Basic definitions

Throughout this chapter we fix an integer $d \geq 0$ and a connected CW-space X with base point x.

1.1 Preliminaries. We shall work in the category of smooth manifolds although all our definitions have topological and piecewise-linear versions. By a manifold we mean a smooth manifold. A manifold M is *closed* if it is compact and $\partial M = \emptyset$.

A topological space Y is *pointed* if every connected component of Y is provided with a base point. The set of base points of Y is denoted $Y_\bullet$.

A *d-dimensional X-manifold* is a pair (a pointed closed oriented d-dimensional manifold M, a map $g\colon M \to X$ such that $g(M_\bullet) = x$). The map g is called the *characteristic map*. A disjoint union of d-dimensional X-manifolds is an X-manifold in the obvious way. An empty set is viewed as a pointed manifold and an X-manifold of any given dimension. We emphasize that by definition, all X-manifolds are closed, oriented, and pointed.

An *X-homeomorphism of X-manifolds* $(M, g) \to (M', g')$ is an orientation preserving diffeomorphism $f\colon M \to M'$ such that $g = g'f$ and $f(M_\bullet) = M'_\bullet$. The equality $g = g'f$ is understood as a coincidence of maps and not just a homotopy.

A *$(d+1)$-dimensional cobordism* is a triple (W, M_0, M_1) where W is a compact oriented $(d+1)$-dimensional manifold and M_0, M_1 are disjoint pointed closed oriented d-dimensional submanifolds of ∂W such that $\partial W = (-M_0) \amalg M_1$. As usual, $-M$ is M with reversed orientation. The manifold W itself is not required to be pointed. Here and below, we use the "outward vector first" convention for the induced orientation on the boundary: at any point of ∂W the given orientation of W is determined by the tuple (a tangent vector directed outward, a basis in the tangent space of ∂W positive with respect to the induced orientation).

An *X-cobordism* is a cobordism (W, M_0, M_1) endowed with a map $g\colon W \to X$ carrying the base points of M_0, M_1 to x; see Figure I.1 where the dots on the components of M_0, M_1 represent the base points. Both the *bottom base* M_0 and the *top base* M_1 of W are considered as X-manifolds with characteristic maps obtained by restricting g. For example, for any X-manifold (M, g), we have the *cylinder X-cobordism*

$$(M \times [0,1], M \times 0, M \times 1, \bar{g}\colon M \times [0,1] \to X),$$

where $\bar{g}$ is the composition of the projection $M \times [0,1] \to M$ with $g\colon M \to X$. It is understood that $M \times [0,1]$ is oriented so that

$$\partial(M \times [0,1]) = (-M \times 0) \amalg (M \times 1).$$

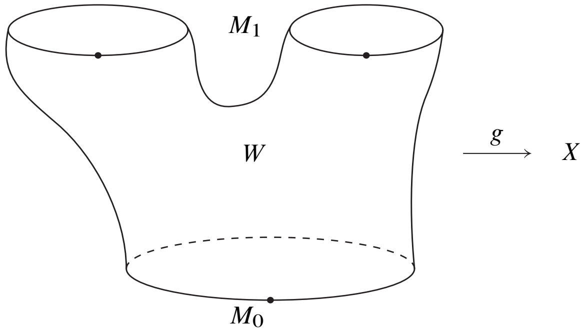


Figure I.1. An X-cobordism (W, M_0, M_1, g).

Disjoint unions of X-cobordisms are X-cobordisms in the obvious way. Two X-cobordisms (W_0, M_0, N, g_0) and (W_1, N', M_1, g_1) can be glued along an arbitrary X-homeomorphism $f: N \to N'$ into a new X-cobordism with bases M_0 and M_1. Here it is essential that $g_0 = g_1 f: N \to X$. An *X-homeomorphism of X-cobordisms*

$$F: (W, M_0, M_1, g) \to (W', M_0', M_1', g') \tag{1.1.a}$$

is an orientation preserving and base point preserving diffeomorphism of triples $(W, M_0, M_1) \to (W', M_0', M_1')$ such that $g = g'F$.

For brevity, we shall often omit the maps to X from the notation for X-manifolds and X-cobordisms.

1.2 Axioms for HQFTs. We define a Homotopy Quantum Field Theory (HQFT) with target X using a version of Atiyah's axioms for a Topological Quantum Field Theory (TQFT); see [Wi1] and [At]. The HQFT will take values in the category of projective K-modules of finite type (= direct summands of K^n with $n = 0, 1, \dots$) and K-linear homomorphisms. One may restrict oneself to the case where K is a field so that projective K-modules of finite type are just finite-dimensional vector spaces over K.

A *$(d+1)$-dimensional Homotopy Quantum Field Theory* (A, τ) *with target* X or, shorter, a *$(d+1)$-dimensional X-HQFT* (A, τ) assigns a projective K-module of finite type A_M to any d-dimensional X-manifold M, a K-isomorphism $f_\#: A_M \to A_{M'}$ to any X-homeomorphism of d-dimensional X-manifolds $f: M \to M'$, and a K-homomorphism $\tau(W): A_{M_0} \to A_{M_1}$ to any $(d+1)$-dimensional X-cobordism (W, M_0, M_1). These modules and homomorphisms should satisfy the following eight axioms.

(1.2.1) For any X-homeomorphisms of d-dimensional X-manifolds $f: M \to M'$, $f': M' \to M''$, we have $(f'f)_\# = f'_\# f_\#$.

(1.2.2) For any disjoint d-dimensional X-manifolds M, N, there is a natural isomorphism $A_{M \amalg N} = A_M \otimes A_N$. Here and below $\otimes = \otimes_K$ is the tensor product over K.

(1.2.3) $A_\emptyset = K$.

(1.2.4) [Topological invariance] For an arbitrary X-homeomorphism (1.1.a) of $(d+1)$-dimensional X-cobordisms, the following diagram commutes:

$$\begin{array}{ccc} A_{(M_0, g|_{M_0})} & \xrightarrow{(F|_{M_0})_\#} & A_{(M'_0, g'|_{M'_0})} \\ \downarrow{\scriptstyle \tau(W,g)} & & \downarrow{\scriptstyle \tau(W',g')} \\ A_{(M_1, g|_{M_1})} & \xrightarrow{(F|_{M_1})_\#} & A_{(M'_1, g'|_{M'_1})}. \end{array}$$

(1.2.5) If a $(d+1)$-dimensional X-cobordism W is a disjoint union of X-cobordisms W_1, W_2, then $\tau(W) = \tau(W_1) \otimes \tau(W_2)$.

(1.2.6) [Gluing axiom] If an X-cobordism (W, M_0, M_1) is obtained from two $(d+1)$-dimensional X-cobordisms (W_0, M_0, N) and (W_1, N', M_1) by gluing along an X-homeomorphism $f: N \to N'$, then

$$\tau(W) = \tau(W_1) \circ f_\# \circ \tau(W_0): A_{M_0} \to A_{M_1}.$$

(1.2.7) For any d-dimensional X-manifold (M, g), we have

$$\tau(M \times [0,1], M \times 0, M \times 1, \bar{g}) = \mathrm{id}: A_M \to A_M,$$

where we identify $M \times 0$ and $M \times 1$ with M in the usual way and where $\bar{g}$ is the composition of the projection $M \times [0,1] \to M$ with $g: M \to X$.

(1.2.8) For any $(d+1)$-dimensional X-cobordism $(W, g: W \to X)$, the homomorphism $\tau(W, g)$ is preserved under homotopies of g constant on ∂W.

Axiom (1.2.2) implies that if a d-dimensional X-manifold M splits as a disjoint union of d-dimensional X-manifolds $M_1, \dots, M_n$, then A_M is canonically isomorphic to the tensor product $\bigotimes_{i=1}^n A_{M_i}$. The naturality condition in Axiom (1.2.2) means that the isomorphism in question is natural with respect to X-homeomorphisms and compatible with permutations of the labels $1, 2, \dots, n$. For a more detailed formulation of the naturality, see [Tu2], p. 21.

As an exercise, the reader may deduce from the axioms of an HQFT that for any X-homeomorphisms of d-dimensional X-manifolds $f, g: M \to M'$, we have $f_\# = g_\#: A_M \to A_{M'}$ provided f is isotopic to g in the class of X-homeomorphisms. In the case where X is aspherical, we shall establish the same claim under the weaker assumption that f and g are isotopic in the class of diffeomorphisms $M \to N$ carrying $M_\bullet$ onto $N_\bullet$ (Lemma 3.3.1).

For each $d \geq 0$, we have the *trivial* $(d+1)$-dimensional X-HQFT (A, τ) defined by $A_M = 0$ for all $M \neq \emptyset$ and $A_\emptyset = K$, $f_\# = 0$ for any X-homeomorphism f of non-empty d-dimensional X-manifolds and $(\mathrm{id}_\emptyset)_\# = \mathrm{id}_K$; $\tau(W) = 0$ for any non-empty $(d+1)$-dimensional X-cobordism W and $\tau(W) = \mathrm{id}_K$ for the empty $(d+1)$-dimensional X-cobordism W. All axioms of an HQFT are straightforward. More interesting examples of HQFTs will be given in Section 2.

If the target space of an HQFT has only one point, then all references to maps into this space are redundant, and we obtain the usual definition of a TQFT for pointed closed d-dimensional manifolds and their cobordisms. Any X-HQFT induces a TQFT by restricting to those X-manifolds and X-cobordisms whose characteristic map is the constant map to $x \in X$.

The definition of an HQFT above was first given in [Tu3]. Note that Axioms (1.2.1) and (1.2.7) are weakened versions of the corresponding axioms in [Tu3]. Two-dimensional HQFTs with simply connected targets were independently and simultaneously introduced in [BT1].

Axioms (1.2.1)–(1.2.7) constitute a special case of the axioms of a TQFT formulated in [Tu2], Chapter III in the language of space-structures. Although we shall not use this language, note that the structures of an X-manifold and an X-cobordism form a cobordism theory in the sense of [Tu2]. Axiom (1.2.8) adds homotopy flavor in this setting.

Another well known approach to TQFTs consists in introducing a monoidal tensor category of d-dimensional manifolds and their cobordisms and defining a $(d+1)$-dimensional TQFT as a monoidal functor from this category to the category of projective K-modules of finite type; see, for instance, [Kock]. This approach is equivalent to Atiyah's axiomatic definition and generalizes to HQFTs, but we will not need it. Quinn [Qu] introduced yet another axiomatic approach to TQFTs; it generalizes to a different set of axioms for HQFTs.

The definition of an HQFT given above can be further generalized by replacing K-modules and K-homomorphisms by objects and morphisms of a symmetric tensor category. Other definitions in this book can be similarly extended, but we shall not pursue this line.

1.3 Properties of HQFTs. Consider a $(d+1)$-dimensional X-HQFT (A, τ). For a compact oriented $(d+1)$-dimensional manifold W with pointed boundary and a map $g: (W, (\partial W)_\bullet) \to (X, x)$, the HQFT (A, τ) yields a K-module $A_{\partial W}$ and a homomorphism $\tau(W, \emptyset, \partial W): K = A_\emptyset \to A_{\partial W}$. Let $\tau(W) \in A_{\partial W}$ be the image of the unity 1_K under this homomorphism. The vector $\tau(W)$ is invariant under homotopy of g constant on ∂W, natural with respect to X-homeomorphisms, and multiplicative under disjoint union. If $\partial W = \emptyset$, then $\tau(W) \in A_\emptyset = K$. In this way, (A, τ) produces a numerical homotopy invariant of maps from closed oriented $(d+1)$-dimensional manifolds to X. By Axiom (1.2.7), if $W = \emptyset$, then $\tau(W) = 1_K$.

We state three properties of (A, τ) that follow from the general theory of TQFTs; see [Tu2], Section III.2. For completeness, we shall give a direct proof in Section 5.

Recall that for a projective K-module of finite type P and a K-linear homomorphism $f: P \to P$, one has a naturally defined *trace* $\mathrm{Tr}(f) \in K$; see, for instance, [Tu2], Appendix 1, or Section 5.1 below. The *dimension* of P is defined by $\mathrm{Dim}\, P = \mathrm{Tr}(\mathrm{id}_P) \in K$, where id_P is the identity automorphism of P. These trace and dimension are multiplicative with respect to the tensor product. If P is a free K-module of finite rank r, then $\mathrm{Tr}(f)$ is the standard trace of f and $\mathrm{Dim}\, P = r \cdot 1_K \in K$.

For a d-dimensional X-manifold M, the *opposite X-manifold* $-M$ is defined as the same manifold with the same map to X but with opposite orientation. A fundamental property of a $(d+1)$-dimensional X-HQFT (A, τ) is the duality $A_{-M} = (A_M)^* = \mathrm{Hom}_K(A_M, K)$. To give a precise statement, recall that for K-modules P, Q, a bilinear pairing $P \otimes Q \to K$ is *nondegenerate* if both adjoint homomorphisms $P \to Q^* = \mathrm{Hom}_K(Q, K)$ and $Q \to P^*$ are isomorphisms.

1.3.1 Lemma. *For a d-dimensional X-manifold M, there is a canonical nondegenerate bilinear pairing $\eta_M: A_M \otimes A_{-M} \to K$. The pairings $\{\eta_M\}_M$ are natural with respect to X-homeomorphisms, multiplicative with respect to disjoint union, and symmetric in the sense that $\eta_{-M} = \eta_M \circ \sigma$ where σ is the standard flip $A_{-M} \otimes A_M \to A_M \otimes A_{-M}$.*

The pairing η_M induces an identification $A_{-M} = A_M^*$ used in the next lemma.

1.3.2 Lemma. *Let $W = (W, M_0, M_1 \amalg M, g)$ be a $(d+1)$-dimensional X-cobordism whose top base is a disjoint union of X-manifolds M_1 and M. Then the tuple $W' = (W, M_0 \amalg -M, M_1, g)$ is also a $(d+1)$-dimensional X-cobordism and the homomorphisms*

$$\tau(W) \in \mathrm{Hom}_K(A_{M_0}, A_{M_1} \otimes A_M) \quad \textit{and} \quad \tau(W') \in \mathrm{Hom}_K(A_{M_0} \otimes A_{-M}, A_{M_1})$$

are obtained from each other via the isomorphisms

$$\mathrm{Hom}(A_{M_0}, A_{M_1} \otimes A_M) = \mathrm{Hom}(A_{M_0} \otimes A_M^*, A_{M_1}) = \mathrm{Hom}(A_{M_0} \otimes A_{-M}, A_{M_1}),$$

where the first isomorphism is standard and the second one is induced by the identification $A_M^* = A_{-M}$.

The cobordisms W and W' are schematically shown in Figure I.2. In this and further figures we omit the mapping to X.

To state the next lemma, we need the notion of a partial trace. Let P, P', Q, Q' be projective K-modules of finite type. For K-homomorphisms $F: P \otimes P' \to Q \otimes Q'$ and $f: Q' \to P'$, the *partial trace of F with respect to f* is the K-homomorphism $\mathrm{Tr}_f(F): P \to Q$ determined by the following property: for any $p \in P, q \in Q^*$,

$$q(\mathrm{Tr}_f(F)(p)) = \mathrm{Tr}(P' \to P', x \mapsto (q \otimes f)F(p \otimes x)),$$

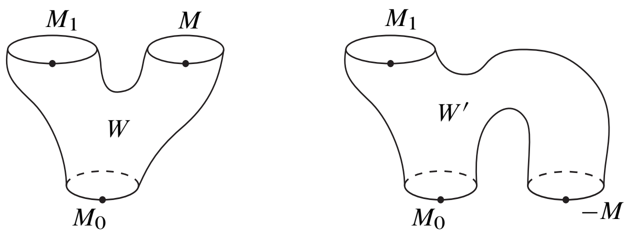


Figure I.2. The X-cobordisms W and W'.

where x runs over P' and $q \otimes f : Q \otimes Q' \to K \otimes P' = P'$ carries $r \otimes r'$ with $r \in Q, r' \in Q'$ to $q(r)f(r') \in P'$. In particular, if $P = Q = K$, then

$$P \otimes P' = P', \quad Q \otimes Q' = Q', \quad \text{and} \quad \mathrm{Tr}_f(F) = \mathrm{Tr}(fF).$$

1.3.3 Lemma. *Let W be a $(d+1)$-dimensional X-cobordism whose bottom (resp. top) base is a disjoint union $M_0 \amalg M_0'$ (resp. $M_1 \amalg M_1'$). Let (V, M_0, M_1) be the X-cobordism obtained from W by gluing M_1' to M_0' along an X-homeomorphism $f: M_1' \to M_0'$; see Figure I.3. Then $\tau(V) = \mathrm{Tr}_{f_\#}(\tau(W)): A_{M_0} \to A_{M_1}$, where $f_\#: A_{M_1'} \to A_{M_0'}$ is the isomorphism induced by f and we view $\tau(W)$ as a homomorphism $A_{M_0} \otimes A_{M_0'} \to A_{M_1} \otimes A_{M_1'}$.*

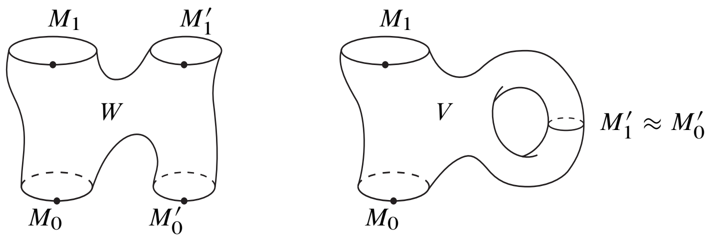


Figure I.3. The X-cobordisms W and V.

Given a d-dimensional X-manifold (M, g), we can apply this lemma to the cylinder $(M \times [0,1], \bar{g})$ and an X-homeomorphism $f: M \to M$. This computes the value of τ on the mapping torus of f to be $\mathrm{Tr}(f_\#: A_M \to A_M)$. In particular, $\tau(M \times S^1, \overline{\overline{g}}) = \mathrm{Dim}\, A_M$, where $\overline{\overline{g}}$ is the composition of the projection $M \times S^1 \to M$ with $g: M \to X$.

1.4 Category of HQFTs. We define a category, $Q_{d+1}(X) = Q_{d+1}(X; K)$, whose objects are $(d+1)$-dimensional HQFTs over the ring K with target X. A morphism $(A, \tau) \to (A', \tau')$ in the category $Q_{d+1}(X)$ is a family of K-isomorphisms $\{\rho_M : A_M \to A'_M\}_M$ (where M runs over all d-dimensional X-manifolds) such that: $\rho_\emptyset = \mathrm{id}_K$, $\rho_{M \amalg N} = \rho_M \otimes \rho_N$ for any disjoint M, N, and the natural square diagrams associated with X-homeomorphisms and X-cobordisms are commutative. Note that all morphisms in the category $Q_{d+1}(X)$ are isomorphisms. Two $(d+1)$-dimensional HQFTs (A,τ), (A',τ') are said to be *isomorphic* if there is a morphism $(A,\tau) \to (A',\tau')$ in the category $Q_{d+1}(X)$.

HQFTs can be pulled back along maps between target spaces. Having a pointed map of pointed connected CW-spaces $f\colon X' \to X$, we can transform any X-HQFT (A,τ) into an X'-HQFT. It suffices to compose the X'-valued characteristic maps with f and to apply (A,τ). This yields a functor $f^*\colon Q_{d+1}(X) \to Q_{d+1}(X')$.

It is easy to deduce from the definitions that the isomorphism classes of $(0+1)$-dimensional X-HQFTs over K correspond bijectively to the isomorphism classes of linear actions of $\pi_1(X)$ on projective K-modules of finite type or, if K is a field, to flat vector bundles over X. Under this correspondence, the invariant τ of a one-dimensional X-manifold $(S^1, g\colon S^1 \to X)$ is equal to the trace of linear endomorphisms determined by the conjugacy class in $\pi_1(X)$ represented by g. We can view high-dimensional X-HQFTs as high-dimensional analogues of flat vector bundles over X.

1.5 Operations on HQFTs. We define several simple operations on $(d+1)$-dimensional HQFTs with target X.

1. The dual $(\overline{A},\bar\tau)$ of an HQFT (A,τ) is defined as follows. For any X-manifold M, set $\overline{A}_M = \mathrm{Hom}_K(A_M, K)$. The action of X-homeomorphisms is obtained by transposition from the one given by (A,τ). Note that each X-cobordism (W, M_0, M_1, g) gives rise to the *opposite X-cobordism* $(-W, M_1, M_0, g)$. We define $\bar\tau(W)\colon A^*_{M_0} \to A^*_{M_1}$ as the transpose of $\tau(-W)\colon A_{M_1} \to A_{M_0}$. The axioms of an HQFT for $(\overline{A},\bar\tau)$ are straightforward.

2. The tensor product $(A\otimes A', \tau\otimes\tau')$ of $(d+1)$-dimensional X-HQFTs (A,τ), (A',τ') is defined by $(A\otimes A')_M = A_M \otimes A'_M$. The action of X-cobordisms and X-homeomorphisms is obtained as the tensor product of the actions provided by (A,τ) and (A',τ'). The axioms of an X-HQFT are straightforward.

3. We define a direct sum $(A\oplus A', \tau\oplus\tau')$ of $(d+1)$-dimensional X-HQFTs (A,τ), (A',τ'). Set $(A\oplus A')_\emptyset = K$. For a non-empty connected d-dimensional X-manifold M, set $(A\oplus A')_M = A_M \oplus A'_M$. This extends to nonconnected M via Axiom (1.2.2). The action of X-homeomorphisms is defined in the obvious way applying $\oplus$ and $\otimes$ to the actions provided by (A,τ) and (A',τ'). Note that for any d-dimensional X-manifold M, we have natural K-linear embeddings

$$i_M\colon A_M \to (A\oplus A')_M, \quad i'_M\colon A'_M \to (A\oplus A')_M$$

and natural K-linear projections

$$p_M\colon (A\oplus A')_M \to A_M, \quad p'_M\colon (A\oplus A')_M \to A'_M$$

such that $p_M i_M = \mathrm{id}$ and $p'_M i'_M = \mathrm{id}$. (If $M=\emptyset$, then $i_M = p_M = \mathrm{id}_K\colon K \to K$.) For a connected $(d+1)$-dimensional X-cobordism (W, M, N), set

$$(\tau\oplus\tau')(W) = i_N\tau(W)p_M + i'_N\tau'(W)p'_M\colon (A\oplus A')_M \to (A\oplus A')_N.$$

This extends to nonconnected W via Axiom (1.2.5). It is not difficult to check that $(A \oplus A', \tau \oplus \tau')$ is an X-HQFT. In particular, for a closed connected oriented $(d+1)$-dimensional manifold W and a map $g: W \to X$,

$$(\tau \oplus \tau')(W, g) = \tau(W, g) + \tau'(W, g).$$

4. HQFTs can be rescaled using additive numerical invariants of X-cobordisms. A $\mathbb{Z}$-valued invariant of $(d+1)$-dimensional X-cobordisms is *additive* if it is preserved under X-homeomorphisms and homotopies of the characteristic map (rel ∂) and is additive under disjoint unions and gluing. A simple example of an additive invariant of a $(d+1)$-dimensional X-cobordism (W, M, N) is provided by the relative Euler characteristic $\chi(W, M) = \chi(W) - \chi(M)$. Given an additive invariant ρ of $(d+1)$-dimensional X-cobordisms and an invertible element k of K, we can transform a $(d+1)$-dimensional X-HQFT (A, τ) into a *k^ρ-rescaled HQFT* which coincides with (A, τ) except that it associates with every $(d+1)$-dimensional X-cobordism W the homomorphism $k^{\rho(W)}\tau(W)$.

Given $k \in K^*$, by *k-rescaling* of a 2-dimensional HQFT with target X, we mean the k^ρ-rescaling, where ρ is the additive invariant of 2-dimensional X-cobordisms defined by

$$\rho(W, M, N) = (\chi(W) + b_0(M) - b_0(N))/2 \in \mathbb{Z}.$$

Here $b_0(M)$ is the number of components of M.

I.2 Cohomological HQFTs and transfer

2.1 Primitive cohomological HQFTs. For each $\theta \in H^{d+1}(X; K^*)$, we define a $(d+1)$-dimensional X-HQFT (A^θ, τ^θ) called the *primitive cohomological HQFT associated with θ*. This HQFT extends to cobordisms the standard evaluation of θ on $(d+1)$-dimensional geometric cycles in X. (Another such extension is provided by the Cheeger–Simons differential characters; see [Turn] for a comparison.)

Fix a singular $(d+1)$-dimensional cocycle on X with values in K^* representing the cohomology class θ. By abuse of notation, we denote this cocycle by the same symbol θ. Let $M = (M, g: M \to X)$ be a d-dimensional X-manifold. Then A_M is a free K-module of rank 1 defined as follows. A d-dimensional singular cycle $a \in C_d(M) = C_d(M; \mathbb{Z})$ is *fundamental* if it represents the fundamental class $[M] \in H_d(M; \mathbb{Z})$ defined as the sum of the fundamental classes of the components of M. Every fundamental cycle $a \in C_d(M)$ determines a generating vector $\langle a \rangle \in A_M = K\langle a \rangle$. If $a, b \in C_d(M)$ are two fundamental cycles, then there is a $(d+1)$-dimensional singular chain c in M such that $\partial c = a - b$. We impose the equality $\langle b \rangle = g^*(\theta)(c)\langle a \rangle$, where $g^*(\theta)$ is the singular $(d+1)$-dimensional cocycle on M obtained by pulling back θ along $g: M \to X$. Note that $g^*(\theta)(c) \in K^*$ does not depend on the choice of c: if $c, c' \in C_{d+1}(M)$ satisfy $\partial c = \partial c'$, then $c - c' = \partial e$ with $e \in C_{d+2}(M)$ and

$$g^*(\theta)(c)(g^*(\theta)(c'))^{-1} = g^*(\theta)(c - c') = g^*(\theta)(\partial e) = g^*(\partial\theta)(e) = 1.$$

It is easy to check that A_M is a well-defined free K-module of rank 1. In particular, if $M = \emptyset$, then $a = 0$ and, by definition, $A_M = K$ and $\langle a \rangle = 1_K \in K$.

An X-homeomorphism of d-dimensional X-manifolds $f: M \to M'$ induces an isomorphism $f_\#: A_M \to A_{M'}$ by $f_\#(\langle a \rangle) = \langle f_*(a) \rangle \in A_{M'}$ for any fundamental cycle $a \in C_d(M)$.

Consider a $(d+1)$-dimensional X-cobordism $(W, M_0, M_1, g: W \to X)$. Pick a singular chain $B \in C_{d+1}(W) = C_{d+1}(W; \mathbb{Z})$ such that $\partial B = b_1 - b_0$, where b_0, b_1 are fundamental cycles in M_0, M_1, respectively. We also require the chain B to be *fundamental* in the sense that its image in $C_{d+1}(W, \partial W; \mathbb{Z})$ represents the fundamental class $[W] \in H_{d+1}(W, \partial W; \mathbb{Z})$ (defined as the sum of the fundamental classes of the components of W). We define $\tau(W): A_{M_0} \to A_{M_1}$ by

$$\tau(W)(\langle b_0 \rangle) = g^*(\theta)(B)\, \langle b_1 \rangle.$$

Let us verify that $\tau(W)$ does not depend on the choice of B. Let $B' \in C_{d+1}(W)$ be another fundamental chain with $\partial B' = b'_1 - b'_0$, where b'_i is a fundamental cycle in M_i for $i = 0, 1$. Pick $c_i \in C_{d+1}(M_i)$ such that $\partial c_i = b_i - b'_i$ for $i = 0, 1$. By definition, $\langle b'_i \rangle = g^*(\theta)(c_i)\langle b_i \rangle$. To see that $\tau(W)$ is well defined it suffices to check the following equality in K^*:

$$g^*(\theta)(c_0)\, g^*(\theta)(B) = g^*(\theta)(c_1)\, g^*(\theta)(B'). \tag{2.1.a}$$

Clearly, $B + c_0 - B' - c_1 \in C_{d+1}(W)$ is a cycle representing 0 in the group $H_{d+1}(W, \partial W; \mathbb{Z})$. Since the inclusion $H_{d+1}(W; \mathbb{Z}) \to H_{d+1}(W, \partial W; \mathbb{Z})$ is injective, the cycle $B + c_0 - B' - c_1$ is a boundary in W. This implies (2.1.a).

We now verify the axioms of an HQFT. Axioms (1.2.1)–(1.2.5) are straightforward. Let us check Axiom (1.2.6). Consider an X-cobordism $(W, M_0, M_1, g: W \to X)$ that is obtained from two $(d+1)$-dimensional X-cobordisms (W_0, M_0, N, g_0) and (W_1, N', M_1, g_1) by gluing along an X-homeomorphism $f: N \to N'$. Let $B_0 \in C_{d+1}(W_0)$ be a fundamental chain with $\partial B_0 = b - b_0$, where b_0, b are fundamental cycles in M_0, N, respectively. Clearly, $f_*(b)$ is a fundamental cycle in N'. Choose a fundamental chain $B_1 \in C_{d+1}(W_1)$ such that $\partial B = b_1 - f_*(b)$, where b_1 is a fundamental cycle in M_1. By definition, the composition $\tau(W_1) \circ f_\# \circ \tau(W_0)$ carries $\langle b_0 \rangle \in A_{M_0}$ to $g_0^*(\theta)(B_0)\, g_1^*(\theta)(B_1)\langle b_1 \rangle$. Observe that the gluing $W_0 \amalg W_1 \to W$ transforms $B_0 + B_1$ into a fundamental chain $B \in C_{d+1}(W)$ with $\partial B = b_1 - b_0$. Clearly, $g^*(\theta)(B) = g_0^*(\theta)(B_0)\, g_1^*(\theta)(B_1)$. Therefore

$$\tau(W)(\langle b_0 \rangle) = g^*(\theta)(B)\langle b_1 \rangle = (\tau(W_1) \circ f_\# \circ \tau(W_0))(\langle b_0 \rangle).$$

As for Axiom (1.2.7), we consider a d-dimensional X-manifold $(M, g: M \to X)$ together with the induced map $\bar{g}: M \times [0,1] \to X$. Choose a fundamental chain

$B \in C_{d+1}(M \times [0,1])$ such that $\partial B = (b \times 1) - (b \times 0)$, where b is a fundamental cycle in M. By definition, the homomorphism $\tau(M \times [0,1], \bar{g})$ carries the generator $\langle b \rangle \in A_M$ into $\bar{g}^*(\theta)(B)\langle b \rangle$. We must prove that $\bar{g}^*(\theta)(B) = 1$. Consider the map $p\colon M \times [0,1] \to M \times S^1$ obtained by the gluing $M \times 0 = M = M \times 1$. It is clear that $p_*(B)$ is a fundamental cycle in $M \times S^1$ and $\bar{g} = \check{g}p$, where $\check{g}\colon M \times S^1 \to X$ is the composition of the projection $M \times S^1 \to M$ with g. Then

$$\bar{g}^*(\theta)(B) = \theta(\bar{g}_*(B)) = \theta(\check{g}_*([M \times S^1])).$$

The homology class $\check{g}_*([M \times S^1]) \in H_{d+1}(X;\mathbb{Z})$ is zero since $\check{g}$ extends to a map $M \times (2\text{-disk}) \to X$. Therefore $\bar{g}^*(\theta)(B) = \theta(0) = 1$.

Let us verify Axiom (1.2.8). Let $(W, g\colon W \to X)$ be a $(d+1)$-dimensional X-cobordism. We should verify that for a fundamental chain $B \in C_{d+1}(W)$, the element $g^*(\theta)(B) \in K^*$ is preserved under any homotopy of g constant on ∂W. Consider the manifold $\widehat{W}$ obtained from $W \times [0,1]$ by contracting to a point each interval $w \times [0,1]$ with $w \in \partial W$. A homotopy of g constant on ∂W is nothing but a map $\hat{g}\colon \widehat{W} \to X$ extending g on $W \times 0 \subset \partial \widehat{W}$. The manifold $\partial \widehat{W}$ is the result of gluing of $W \times 0$ and $W \times 1$ along $\partial W \times 0 = \partial W = \partial W \times 1$. The chain $(B \times 0) - (B \times 1)$ in $\partial \widehat{W}$ is a cycle representing the fundamental class of $\partial \widehat{W}$. Therefore this cycle bounds a singular chain in $\widehat{W}$. This implies that $\hat{g}^*(\theta)(B \times 0) = \hat{g}^*(\theta)(B \times 1)$, the equality we need. This completes the verification of the axioms and shows that $(A, \tau) = (A^\theta, \tau^\theta)$ is an HQFT. Note that by definition, for a closed oriented $(d+1)$-dimensional manifold W and a map $g\colon W \to X$,

$$\tau^\theta(W, g) = g^*(\theta)([W]) = \theta(g_*([W])) \in K^*.$$

It is easy to check that the primitive cohomological HQFTs arising as above from different singular cocycles representing $\theta \in H^{d+1}(X; K^*)$ are isomorphic. Therefore the isomorphism class of the X-HQFT (A^θ, τ^θ) depends only on θ. Note that the base points play no role in the construction of (A^θ, τ^θ).

For $\theta = 0 \in H^{d+1}(X; K^*)$, the HQFT (A^θ, τ^θ) assigns K to all d-dimensional X-manifolds and assigns $\mathrm{id}_K\colon K \to K$ to all X-homeomorphisms and all $(d+1)$-dimensional X-cobordisms. As an exercise, the reader may verify that

$$(A^{\theta+\theta'}, \tau^{\theta+\theta'}) = (A^\theta, \tau^\theta) \otimes (A^{\theta'}, \tau^{\theta'})$$

and $(A^{-\theta}, \tau^{-\theta}) = ((A^\theta)^*, (\tau^\theta)^*)$ for any $\theta, \theta' \in H^{d+1}(X; K^*)$.

2.2 Transfer. The transfer is a push-forward operation on HQFTs associated with a finite-sheeted (unramified) covering $p\colon E \to X$. Contracting the finite set $p^{-1}(x) \subset E$ into a point, we obtain a pointed connected CW-space $E' = E/p^{-1}(x)$. From any $(d+1)$-dimensional HQFT (A, τ) with target E' we shall derive a $(d+1)$-dimensional HQFT $(\tilde{A}, \tilde{\tau})$ with target X. It is called the *transfer* of (A, τ).

Let $q\colon E \to E'$ be the projection. For a d-dimensional X-manifold $(M, g\colon M \to X)$, consider all lifts of g to E, i.e., all maps $\tilde{g}\colon M \to E$ such that $p\tilde{g} = g$. The set of such $\tilde{g}$ is finite (possibly empty), and each pair $(M, q\tilde{g})$ is an E'-manifold. Set

$$\tilde{A}_M = \bigoplus_{\substack{\tilde{g}\colon M \to E \\ p\tilde{g}=g}} A_{(M,q\tilde{g})}.$$

An X-homeomorphism of d-dimensional X-manifolds $f\colon (M,g) \to (M',g')$ induces a K-isomorphism $\tilde{A}_{(M,g)} \to \tilde{A}_{(M',g')}$ as follows. Since $g = g'f$, any lift $\tilde{g}\colon M \to E$ of g induces a lift $\tilde{g}f^{-1}\colon M' \to E$ of g'. The HQFT (A,τ) gives an isomorphism $f_\#\colon A_{(M,q\tilde{g})} \to A_{(M',q\tilde{g}f^{-1})}$. The direct sum of these isomorphisms over all $\tilde{g}$ is the desired isomorphism $f_\#\colon \tilde{A}_{(M,g)} \to \tilde{A}_{(M',g')}$.

Consider a $(d+1)$-dimensional X-cobordism $(W, M_0, M_1, g\colon W \to X)$. As above, the set of maps $\tilde{g}\colon W \to E$ such that $p\tilde{g} = g$ is finite (possibly empty). Restricting such $\tilde{g}$ to M_0, M_1 we obtain certain lifts, $\tilde{g}_0$, $\tilde{g}_1$, of the maps $g_0 = g|_{M_0}\colon M_0 \to X$ and $g_1 = g|_{M_1}\colon M_1 \to X$. The HQFT (A,τ) gives a homomorphism

$$\tau(W, q\tilde{g})\colon A_{(M_0,q\tilde{g}_0)} \to A_{(M_1,q\tilde{g}_1)}.$$

The sum of the latter homomorphisms over all lifts $\tilde{g}$ of g defines a homomorphism $\tilde{\tau}(W,g)\colon \tilde{A}_{M_0} \to \tilde{A}_{M_1}$. (Warning: if W has components disjoint from M_0, then a lift of g_0 to E may extend to several lifts of g so that the sum in question is in general not a direct sum.) It is easy to verify that $(\tilde{A}, \tilde{\tau})$ is a $(d+1)$-dimensional X-HQFT. In particular, for a closed oriented $(d+1)$-dimensional manifold W and a map $g\colon W \to X$,

$$\tilde{\tau}(W,g) = \sum_{\substack{\tilde{g}\colon W \to E \\ p\tilde{g}=g}} \tau(W, q\tilde{g}) \in K.$$

Observe that $H^{d+1}(E';K^*) = H^{d+1}(E;K^*)$ for $d \geq 1$. Therefore for any $\theta \in H^{d+1}(E;K^*)$ with $d \geq 1$, we have a primitive cohomological HQFT (A^θ, τ^θ) with target E'. Its transfer $(\tilde{A}^\theta, \tilde{\tau}^\theta)$ to X is an X-HQFT called the *cohomological HQFT associated with* θ. By definition, for a d-dimensional X-manifold $(M, g\colon M \to X)$, the K-module $\tilde{A}^\theta_M$ is a free K-module of rank equal to the number of lifts of g to E. For a closed oriented $(d+1)$-dimensional manifold W and a map $g\colon W \to X$,

$$\tilde{\tau}^\theta(W,g) = \sum_{\substack{\tilde{g}\colon W \to E \\ p\tilde{g}=g}} \tilde{g}^*(\theta)([W]) \ \in K.$$

I.3 Aspherical targets

We study here the case where the target CW-space X is aspherical, i.e., X is an Eilenberg–MacLane space of type $K(G,1)$ for a group G. Then X is determined by G

up to homotopy equivalence so that there is a reasonable hope to describe X-HQFTs in algebraic terms. The asphericity of X implies a number of technical simplifications. In particular, as we shall see, X-HQFTs can be applied to homotopy classes of characteristic maps rather than to maps themselves.

In this section X is an aspherical connected CW-space with base point x and $G = \pi_1(X, x)$. We begin with examples of X-HQFTs.

3.1 Cohomological HQFTs. Every $\theta \in H^{d+1}(X; K^*) = H^{d+1}(G; K^*)$ gives rise to a $(d+1)$-dimensional X-HQFT (A^θ, τ^θ) (see Section 2.1). More generally, consider a subgroup $H \subset G$ of finite index and the associated covering $E \to X$ with $\pi_1(E) = H$. By Section 2.2, any $\theta \in H^{d+1}(H; K^*) = H^{d+1}(E; K^*)$ with $d \geq 1$ yields a cohomological $(d+1)$-dimensional X-HQFT. It is denoted $(A^{G,H,\theta}, \tau^{G,H,\theta})$.

For $d = 1$ and $k \in K^*$, denote by $(A^{G,H,\theta}, \tau^{G,H,\theta,k})$ be the 2-dimensional X-HQFT obtained from $(A^{G,H,\theta}, \tau^{G,H,\theta})$ by the k-rescaling introduced in Section 1.5.4, p. 8. We call $(A^{G,H,\theta}, \tau^{G,H,\theta,k})$ *a rescaled cohomological X-HQFT*. A two-dimensional X-HQFT is *semi-cohomological* if it is isomorphic to a finite direct sum of rescaled cohomological X-HQFTs.

3.2 Homotopy invariance of HQFTs. We now explain that the modules and homomorphisms produced by a $(d+1)$-dimensional X-HQFT (A, τ) depend only on the homotopy classes of the characteristic maps. The key point is the following property of (A, τ).

(3.2.1) For any d-dimensional X-manifold $(M, g\colon M \to X)$ and any map $F\colon M \times [0,1] \to X$ such that $F|_{M\times 0} = F|_{M\times 1} = g$ and $F(M_\bullet \times [0,1]) = x$,

$$\tau(M \times [0,1], M \times 0, M \times 1, F) = \mathrm{id}\colon A_{M,g} \to A_{M,g}.$$

This directly follows from Axioms (1.2.7) and (1.2.8) since F is homotopic to $\bar{g}$ rel $M \times \{0,1\}$. The latter is a corollary of the asphericity of X.

To proceed, we need the following notation. For a pointed space Y, denote the set of homotopy classes of maps $Y \to X$ by $[Y, X]$. (It is understood that the maps carry $Y_\bullet$ to the base point $x \in X$ and the homotopies are constant on $Y_\bullet$.) We introduce *homotopy X-manifolds* similarly to X-manifolds but using homotopy classes of maps to X rather than maps themselves. Thus, a homotopy X-manifold is a pair (a pointed closed oriented d-dimensional manifold M, an element $\mathfrak{G}$ of $[M, X]$). For a homotopy X-manifold $(M, \mathfrak{G})$, we define a K-module $A_{(M,\mathfrak{G})}$ as follows. Any homotopy $h\colon M \times [0,1] \to X$ between two representatives $g, g'\colon M \to X$ of $\mathfrak{G}$ turns $M \times [0,1]$ into an X-cobordism. Applying τ, we obtain a homomorphism $\tau_h\colon A_{(M,g)} \to A_{(M,g')}$. By (3.2.1), τ_h is the identity homomorphism if $g = g'$. Axiom (1.2.6) implies that $\tau_{hh'} = \tau_{h'}\tau_h$ for any composable homotopies h, h'. Therefore $\tau_h\colon A_{(M,g)} \to A_{(M,g')}$ is an isomorphism for all h depending only on g, g' and independent of the choice of h. We identify the modules $\{A_{(M,g)}\}_{g\in\mathfrak{G}}$ along the isomorphisms $\{\tau_h\}_h$. This gives a projective K-module of finite type $A_{(M,\mathfrak{G})}$ depending only on $(M, \mathfrak{G})$ and canonically isomorphic to each $A_{(M,g)}$ with $g \in \mathfrak{G}$.

A *homotopy X-homeomorphism* of homotopy X-manifolds $(M, \mathfrak{G}) \to (M', \mathfrak{G}')$ is an orientation preserving diffeomorphism $f: M \to M'$ such that $f(M_\bullet) = M'_\bullet$ and $\mathfrak{G} = \mathfrak{G}' f$. One can think of f as of an X-homeomorphism $(M, g) \to (M', g')$ for appropriate representatives $g \in \mathfrak{G}$ and $g' \in \mathfrak{G}'$. Namely, $g': M' \to X$ can be any map representing $\mathfrak{G}'$ and $g = g' f$. For a homotopy X-homeomorphism $f: (M, \mathfrak{G}) \to (M', \mathfrak{G}')$, we define $f_\#: A_{(M,\mathfrak{G})} \to A_{(M',\mathfrak{G}')}$ as the composition

$$A_{(M,\mathfrak{G})} \xrightarrow{=} A_{(M,g'f)} \xrightarrow{f_\#} A_{(M',g')} \xrightarrow{=} A_{(M',\mathfrak{G}')},$$

where $g': M' \to X$ is a representative of $\mathfrak{G}'$ and the first and third homomorphisms are the canonical identifications. We claim that this composition does not depend on the choice of $g' \in \mathfrak{G}'$. Indeed, let $g'': M' \to X$ be another representative of $\mathfrak{G}'$. A homotopy h between g' and g'' induces a homotopy $h_f = h \circ (f \times \mathrm{id}_{[0,1]})$ between $g'f$ and $g''f$. By Axiom (1.2.4), the diagram

$$\begin{array}{ccc} A_{(M,g'f)} & \xrightarrow{f_\#} & A_{(M',g')} \\ \downarrow{\scriptstyle \tau_{h_f}} & & \downarrow{\scriptstyle \tau_h} \\ A_{(M,g''f)} & \xrightarrow{f_\#} & A_{(M',g'')} \end{array}$$

is commutative. This implies our claim.

We define a *homotopy X-cobordism* as a cobordism (W, M_0, M_1) endowed with a homotopy class of maps $W \to X$. The maps here should carry the base points of M_0 and M_1 into x and the homotopies should be constant on the base points; we denote the set of the corresponding homotopy classes by $[W, X]$. For a $(d+1)$-dimensional homotopy X-cobordism $(W, M_0, M_1, \mathfrak{G} \in [W, X])$ we define a homomorphism $\tau(W, \mathfrak{G}): A_{(M_0,\mathfrak{G}_0)} \to A_{(M_1,\mathfrak{G}_1)}$ where $\mathfrak{G}_j = \mathfrak{G}|_{M_j} \in [M_j, X]$ for $j = 0, 1$. Let $g: W \to X$ be a map representing $\mathfrak{G}$. Set $g_j = g|_{M_j} \in \mathfrak{G}_j$ for $j = 0, 1$. We define $\tau(W, \mathfrak{G})$ as the composition

$$A_{(M_0,\mathfrak{G}_0)} \xrightarrow{=} A_{(M_0,g_0)} \xrightarrow{\tau(W,g)} A_{(M_1,g_1)} \xrightarrow{=} A_{(M_1,\mathfrak{G}_1)},$$

where the first and third homomorphisms are the canonical identifications and the second homomorphism is determined by the X-cobordism (W, g). We claim that $\tau(W, \mathfrak{G})$ does not depend on the choice of g in $\mathfrak{G}$. Let $g': W \to X$ be another representative of $\mathfrak{G}$ and let h be a homotopy between g and g'. We must prove the commutativity of the diagram

$$\begin{array}{ccc} A_{(M_0,g_0)} & \xrightarrow{\tau(W,g)} & A_{(M_1,g_1)} \\ \downarrow{\scriptstyle \tau_{h_0}} & & \downarrow{\scriptstyle \tau_{h_1}} \\ A_{(M_0,g'_0)} & \xrightarrow{\tau(W,g')} & A_{(M_1,g'_1)} \end{array} \tag{3.2.a}$$

where $g'_j = g'|_{M_j}$ and h_j is the restriction of h to M_j for $j = 0, 1$. Consider the map $\bar{h}_1 \colon M_1 \times [0,1] \to X$ defined by $\bar{h}_1(a,t) = h_1(a, 1-t)$ for $a \in M_1, t \in [0,1]$. Let (W', M_0, M_1) be the X-cobordism obtained by gluing the X-cobordisms

$$(M_0 \times [0,1], h_0), \quad (W, g'), \quad \text{and} \quad (M_1 \times [0,1], \bar{h}_1)$$

along $M_0 \times 1 = M_0 \subset \partial W$ and $M_1 \times 0 = M_1 \subset \partial W$. By Axiom (1.2.6),

$$\tau(W') = \tau_{\bar{h}_1} \tau(W, g')\, \tau_{h_0} = \tau_{h_1}^{-1} \tau(W, g')\, \tau_{h_0}.$$

On the other hand, it is clear that W' is just the same cobordism W with another characteristic map to X. Moreover, the homotopy h of g to g' induces a homotopy of that characteristic map to g (rel ∂W). By Axiom (1.2.8), $\tau(W') = \tau(W, g)$. Hence diagram (3.2.a) is commutative.

These constructions show that studying HQFTs with aspherical targets, we do not need to distinguish between characteristic maps and their homotopy classes. In the sequel, for aspherical X, we make no distinction between X-manifolds (resp. X-cobordisms, X-homeomorphisms) and homotopy X-manifolds (resp. homotopy X-cobordisms, homotopy X-homeomorphisms).

3.3 Action of isotopic X-homeomorphisms. The following lemma shows that isotopic X-homeomorphisms of d-dimensional X-manifolds $M \to N$ induce the same homomorphisms $A_M \to A_N$ for any $(d+1)$-dimensional X-HQFT (A, τ). Note that this lemma applies only to HQFTs with aspherical target and that, in accordance with the conventions above, by X-manifolds and X-homeomorphisms we mean homotopic X-manifolds and homotopic X-homeomorphisms. (In the sequel, we shall not stress this anymore.)

3.3.1 Lemma. *Let $f, f' \colon M \to N$ be X-homeomorphisms of d-dimensional X-manifolds isotopic in the class of diffeomorphisms $M \to N$ carrying $M_\bullet$ onto $N_\bullet$. Then $f_\# = f'_\# \colon A_M \to A_N$ for any $(d+1)$-dimensional X-HQFT (A, τ).*

Proof. Set $I = [0,1]$. Let $F \colon M \times I \to N \times I$ be an isotopy between f and f' such that $F(M_\bullet \times I) = N_\bullet \times I$. Pick a representative $g \colon N \to X$ of the given homotopy class of maps $N \to X$. Let $\bar{g} \colon N \times I \to X$ be the composition of the projection $N \times I \to N$ with g. Then $W' = (N \times I, \bar{g})$ is an X-cobordism between two copies of (N, g) and $W = (M \times I, \bar{g}F)$ is an X-cobordism between (M, gf) and (M, gf'). By Axiom (1.2.4), the following diagram is commutative:

$$\begin{array}{ccc} A_{(M,gf)} & \xrightarrow{f_\#} & A_{(N,g)} \\ {\scriptstyle \tau(W, \bar{g}F)} \downarrow & & \downarrow {\scriptstyle \tau(W', \bar{g})V} \\ A_{(M,gf')} & \xrightarrow{f'_\#} & A_{(N,g)}. \end{array}$$

By Axiom (1.2.7), the right vertical arrow is the identity homomorphism. The equalities $\bar{g}F(M_\bullet \times I) = \bar{g}(N_\bullet \times I) = g(N_\bullet) = \{x\}$ show that the left vertical arrow is precisely the isomorphism used in the previous subsection to identify the K-modules $A_{(M,gf)}$ and $A_{(M,gf')}$ in order to obtain the module associated with the given homotopy class of maps $M \to X$. Therefore $f_\# = f'_\# \colon A_M \to A_N$. □

3.4 Independence of the base point. We show that the category of X-HQFTs is essentially independent of the choice of the base point $x \in X$. To stress the role of x in the definitions above, we use in this subsection the terms (X, x)-manifolds and (X, x)-cobordisms for X-manifolds and X-cobordisms, respectively. We can transport X-HQFTs along paths in X as follows. Let $\beta \colon [0,1] \to X$ be a path in X connecting the points $y = \beta(0)$ and $z = \beta(1)$. For any d-dimensional (X, y)-manifold $(M, g \colon M \to X)$, consider a map $g^\beta : M \times [0,1] \to X$ such that

$(*)$ $g^\beta|_{M\times 0} = g$ and $g^\beta(m \times t) = \beta(t)$ for all $m \in M_\bullet$ and all $t \in [0,1]$.

The existence of g^β follows from the fact that $(M \times 0) \cup (M_\bullet \times [0,1])$ is a deformation retract of $M \times [0,1]$. Any two maps g^β satisfying $(*)$ are homotopic in the class of maps satisfying $(*)$. Restricting g^β to $M = M \times 1$ we obtain a (X, z)-manifold $M^\beta = (M, g^\beta|_{M\times 1})$. A similar construction transforms a (X, y)-cobordism W into an (X, z)-cobordism W^β. Now, every $(d+1)$-dimensional HQFT (A, τ) with target (X, z) gives rise to an HQFT $({}^\beta A, {}^\beta\tau)$ with target (X, y) by $({}^\beta A)_M = A_{M^\beta}$ and $({}^\beta\tau)(W) = \tau(W^\beta)$. The action of X-homeomorphisms is defined similarly. Clearly, $({}^{\beta\beta'}A, {}^{\beta\beta'}\tau) = ({}^\beta({}^{\beta'}A), {}^\beta({}^{\beta'}\tau))$ for any paths $\beta, \beta' \colon [0,1] \to X$ with $\beta(1) = \beta'(0)$. If two paths β, δ are homotopic rel $\{0,1\}$, then $({}^\beta A, {}^\beta\tau) = ({}^\delta A, {}^\delta\tau)$.

Transporting HQFTs along paths, one can show that the category of HQFTs $Q_{d+1}(X)$ does not depend on the choice of the base point of X up to equivalence of categories. For a more precise statement; see Exercise 6 in Appendix 1.

3.5 Remarks. 1. For a connected closed oriented d-dimensional manifold M with base point m, the set $[M, X]$ can be identified with the set of homomorphisms from $\pi_1(M, m)$ to $G = \pi_1(X, x)$. Thus, a $(d+1)$-dimensional X-HQFT (A, τ) assigns a K-module $A_{(M,g)}$ to each such homomorphism g. One can reformulate the axioms of an X-HQFT in terms of group homomorphisms rather than maps to X. In particular, the invariant τ of a connected $(d+1)$-dimensional X-cobordism W with empty bases depends only on the associated conjugacy class of homomorphisms $\pi_1(W) \to G$. For cobordisms with nonempty bases, one has to involve the fundamental groupoids determined by the base points of the boundary components.

2. It is useful to keep in mind the well known connection between maps to $X = K(G, 1)$ and principal G-bundles. A *principal G-bundle* over a space W is a regular covering $\widetilde{W} \to W$ with group of automorphisms G. One calls $\widetilde{W}$ the *total space* and W the *base* of the bundle. Two principal G-bundles over W are *isomorphic* if there is a G-equivariant homeomorphism of their total spaces inducing the identity map

$W \to W$. For example, the universal covering $\widetilde{X} \to X$ is a principal G-bundle. Any map $f : W \to X$ induces a principal G-bundle over W by pulling back the universal covering $\widetilde{X} \to X$ along f. If W is a CW-complex, then this establishes a bijective correspondence between free homotopy classes of maps $W \to X$ and isomorphism classes of principal G-bundles over W.

3. The space $X = K(G, 1)$ is the first building block in the Postnikov system of any connected CW-complex Y with fundamental group G. In particular, we have a mapping $f : Y \to X$ inducing the identity in π_1. Any X-HQFT induces a Y-HQFT by pulling back along f.

4. The constructions of Section 3.4 define a left action of $\pi_1(X, x)$ on $(d + 1)$-dimensional X-HQFTs. On the level of isomorphism classes of HQFTs this action is trivial: for any X-HQFT (A, τ) and any $\beta \in \pi_1(X, x)$, the X-HQFT $({}^{\beta}A, {}^{\beta}\tau)$ is isomorphic to (A, τ). An isomorphism is given by the system of K-isomorphisms $\{\tau(M \times [0, 1], g^{\beta}) : A_M \to A_{M^{\beta}} = ({}^{\beta}A)_M\}_M$, where M runs over all d-dimensional X-manifolds.

I.4 Hermitian and unitary HQFTs

In this section we assume that the ground ring K is endowed with a ring involution $K \to K, k \mapsto \bar{k}$.

4.1 Hermitian and unitary structures on HQFTs. A *Hermitian structure* on a $(d + 1)$-dimensional X-HQFT (A, τ) assigns to each d-dimensional X-manifold M a nondegenerate Hermitian pairing $\langle \cdot , \cdot \rangle_M : A_M \times A_M \to K$ such that

(4.1.1) the pairing $\langle \cdot , \cdot \rangle_M$ is natural with respect to X-homeomorphisms and multiplicative with respect to disjoint union; for $M = \emptyset$ the pairing $\langle \cdot , \cdot \rangle_M$ on $A_M = K$ is determined by the unit (1×1)-matrix;

(4.1.2) for any $(d + 1)$-dimensional X-cobordism $(W, M_0, M_1, g : W \to X)$ and any vectors $a \in A_{M_0}, b \in A_{M_1}$,

$$\langle \tau(W, g)(a), b \rangle_{M_1} = \langle a, \tau(-W, g)(b) \rangle_{M_0}. \tag{4.1.a}$$

An HQFT with Hermitian structure is called a *Hermitian HQFT*. If $K = \mathbb{C}$ with complex conjugation and the Hermitian form $\langle \cdot , \cdot \rangle_M$ is positive definite for every M, then the Hermitian HQFT is *unitary*.

Note two simple properties of a Hermitian $(d+1)$-dimensional HQFT (A, τ). First, if W is a closed oriented $(d + 1)$-dimensional X-manifold, then $\tau(-W, g) = \overline{\tau(W, g)}$ for any map $g : W \to X$. Secondly, the action of X-self-homeomorphisms of a d-dimensional X-manifold M on A_M preserves the Hermitian form $\langle \cdot , \cdot \rangle_M$. For a unitary HQFT, we obtain a unitary action.

We define a category, $HQ_{d+1}(X)$ (resp. $UQ_{d+1}(X)$) whose objects are $(d+1)$-dimensional Hermitian (resp. unitary) X-HQFTs. The morphisms in this category are defined as in Section 1.4 with additional requirement that the isomorphisms $\{\rho_M\}_M$ preserve the Hermitian pairings.

4.2 Example. The construction of cohomological HQFTs can be refined to yield Hermitian HQFTs. Consider the multiplicative group

$$S = \{k \in K^* \mid k\bar{k} = 1\} \subset K^*.$$

For $\theta \in H^{d+1}(X; S)$, the primitive cohomological $(d+1)$-dimensional X-HQFT (A^θ, τ^θ) defined in Section 2.1 can be provided with a Hermitian structure as follows. For a d-dimensional X-manifold M and a fundamental cycle $a \in C_d(M)$, set $\langle\langle a\rangle, \langle a\rangle\rangle_M = 1 \in K$, where $\langle a\rangle \in A_M$ is the generating vector represented by a. This extends by skew-linearity to a Hermitian form on $A_M = K\langle a\rangle$ independent of the choice of a. This form satisfies (4.1.1) and (4.1.2). The key point is that $\Theta(-B) = (\Theta(B))^{-1} = \overline{\Theta(B)}$ for any singular $(d+1)$-dimensional chain B in X and any singular cochain $\Theta \in C^{d+1}(X; S)$. The isomorphism class of the Hermitian HQFT (A^θ, τ^θ) depends only on θ. If $K = \mathbb{C}$, then $S = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the Hermitian HQFT (A^θ, τ^θ) is unitary.

4.3 Remarks. 1. Direct sums, tensor products, and transfers of Hermitian (resp. unitary) HQFTs are Hermitian (resp. unitary) HQFTs. Combining with Example 4.2, we obtain a wide class of Hermitian and unitary HQFTs.

2. The isomorphism classes of 1-dimensional unitary X-HQFTs correspond bijectively to the conjugacy classes of homomorphisms from $\pi_1(X)$ to the complex unitary groups.

3. For a d-dimensional X-manifold $M = (M, g)$ and an X-homeomorphism $f: M \to M$, the mapping torus W_f of f is a $(d+1)$-dimensional closed orientable manifold. The map g induces a map $\hat{g}: W_f \to X$ in the obvious way. Lemma 1.3.3 implies that $|\tau(W_f, \hat{g})| \leq \dim_{\mathbb{C}} A_M$ for any unitary $(d+1)$-dimensional HQFT (A, τ) with target X.

I.5 Proof of Lemmas 1.3.1–1.3.3

We begin with an algebraic lemma.

5.1 Lemma. *Let P, Q be K-modules. Let $\eta: P \otimes Q \to K$ and $\eta^-: K \to Q \otimes P$ be K-homomorphisms such that*

$$(\mathrm{id}_Q \otimes \eta)(\eta^- \otimes \mathrm{id}_Q) = \mathrm{id}_Q \tag{5.1.a}$$

and

$$(\eta \otimes \mathrm{id}_P)(\mathrm{id}_P \otimes \eta^-) = \mathrm{id}_P. \tag{5.1.b}$$

Then the pairing η *is nondegenerate and for any homomorphism* $f: P \to P$ *and any finite expansion* $\eta^-(1_K) = \sum_i q_i \otimes p_i$ *with* $q_i \in Q$, $p_i \in P$,

$$\mathrm{Tr}(f) = \sum_i \eta(f(p_i), q_i) \in K. \tag{5.1.c}$$

Proof. Denote the homomorphisms $Q \to P^*$, $P \to Q^*$ adjoint to η and the homomorphisms $P^* \to Q$, $Q^* \to P$ adjoint to η^- by h_1, h_2, h_3, h_4 respectively. Formula (5.1.a) implies that $h_3 h_1 = \mathrm{id}_Q$. Indeed, if $\eta^-(1_K) = \sum_i q_i \otimes p_i$ with $q_i \in Q$, $p_i \in P$, then (5.1.a) indicates that for any $q \in Q$,

$$\sum_i \eta(p_i, q) q_i = q. \tag{5.1.d}$$

The homomorphism h_1 carries q into the linear functional $p \mapsto \eta(p, q)$ on P and the homomorphism h_3 carries $h_1(q)$ to

$$\sum_i h_1(q)(p_i) q_i = \sum_i \eta(p_i, q) q_i = q.$$

A similar argument deduces from (5.1.a) that $h_2 h_4 = \mathrm{id}_{Q^*}$. Formula (5.1.b) similarly implies that $h_1 h_3 = \mathrm{id}_{P^*}$ and $h_4 h_2 = \mathrm{id}_P$. Hence h_1, h_2 are isomorphisms and the pairing η is nondegenerate.

To prove (5.1.c), recall the definition of the trace Tr following [Tu2], Appendix I. Let R be a K-module such that $P \oplus R$ is a free K-module of finite rank. Consider the endomorphism $f \oplus 0$ of $P \oplus R$, where 0 is the zero endomorphism of R. Set $\mathrm{Tr}(f) = \mathrm{Tr}(f \oplus 0)$, where the right-hand side is the usual trace of the matrix of $f \oplus 0$ with respect to a basis of $P \oplus R$. The trace $\mathrm{Tr}(f)$ does not depend on the choice of R. It is straightforward to check that the evaluation form $\eta_R: R \otimes R^* \to K$ and the co-evaluation $(\eta_R)^*: K \to R^* \otimes R$ satisfy (5.1.a) and (5.1.b). Replacing η and η^- with $\eta \oplus \eta_R$ and $\eta^- \oplus (\eta_R)^*$, respectively, we can reduce (5.1.c) to the case where P is a free K-module of finite rank. Then so is $Q \cong P^*$.

It is obvious that the right-hand side of (5.1.c) does not depend on the choice of the expansion $\eta^-(1_K) = \sum_i q_i \otimes p_i$. We choose one such expansion as follows. Let $q_1, \dots, q_n$ be a basis of Q. We can uniquely expand $\eta^-(1_K) = \sum_{i=1}^n q_i \otimes p_i$ with $p_1, \dots, p_n \in P$. Applying (5.1.d) to $q = q_j$, we obtain that $\eta(p_i, q_j) = \delta_i^j$, where δ is the Kronecker delta. Since η is nondegenerate, the vectors $p_1, \dots, p_n$ form a basis of P. Let $\{f_{i,j}\}$ be the matrix of f with respect to this basis so that $f(p_i) = \sum_j f_{i,j} p_j$ for $i = 1, \dots, n$. Then

$$\sum_i \eta(f(p_i), q_i) = \sum_{i,j} f_{i,j}\, \eta(p_j, q_i) = \sum_{i,j} f_{i,j}\, \delta_i^j = \sum_i f_{i,i} = \mathrm{Tr}(f). \qquad \square$$

5.2 Proof of Lemma 1.3.1. Let $(M, g: M \to X)$ be a d-dimensional X-manifold. Set $P = A_M$ and $Q = A_{-M}$. Denote by C the cylinder $(M \times [0,1], \bar{g})$ as in Axiom (1.2.7). Clearly, $\partial C = (-M) \cup M$. Consider the $(d+1)$-dimensional X-cobordisms $(C, \emptyset, \partial C)$ and $(C, -\partial C, \emptyset)$ schematically shown in Figure I.4. The corresponding operators τ are homomorphisms $P \otimes Q \to K$ and $K \to Q \otimes P$. Denote them by η_M and η_M^-, respectively.

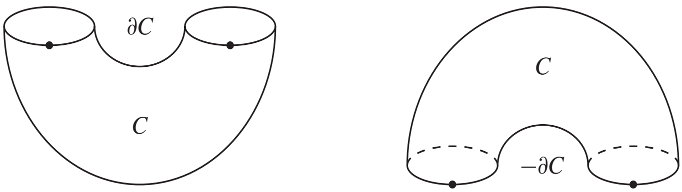


Figure I.4. The X-cobordisms $(C, \emptyset, \partial C)$ and $(C, -\partial C, \emptyset)$.

We claim that the pairings $\{\eta_M\}_M$ satisfy the conditions of the lemma. Indeed, it follows from the axioms of an HQFT that η_M is natural with respect to X-homeomorphisms and multiplicative with respect to disjoint union. Let us verify that $\eta_{-M} = \eta_M \circ \sigma$. Consider the map $f: M \times [0,1] \to M \times [0,1]$ defined by $f(m,t) = (m, 1-t)$ for all $m \in M$, $t \in [0,1]$. It is clear that f is an X-homeomorphism of the X-cobordism

$$(M \times [0,1], (M \times 0) \cup (-M \times 1), \emptyset, \bar{g})$$

onto the X-cobordism

$$((-M) \times [0,1], (-M \times 0) \cup (M \times 1), \emptyset, \bar{g}).$$

The homomorphism induced by the restriction of f to the bottom base

$$f_\#: P \otimes Q = A_{(M \times 0) \cup (-M \times 1)} \to A_{(-M \times 0) \cup (M \times 1)} = Q \otimes P$$

is just the flip $\sigma: P \otimes Q \to Q \otimes P$ (this is a part of the "naturality" requirement in Axiom (1.2.2)). The restriction of f to the top base is $\mathrm{id}_\emptyset$ and the induced automorphism of $A_\emptyset = K$ is the identity, as easily follows from Axiom (1.2.1). Now, Axiom (1.2.4) implies that $\eta_{-M} \circ \sigma = \eta_M$. Therefore $\eta_{-M} = \eta_M \circ \sigma$. Similarly, $\eta_{-M}^- = \sigma \circ \eta_M^-$.

It remains to prove the nondegeneracy of η_M. We shall prove that η_M and η_M^- satisfy the conditions of Lemma 5.1. Let us take four copies C_1, C_2, C_3, C_4 of $C = (M \times [0,1], \bar{g})$. Clearly, $\partial C_i = (-M_i) \amalg M_i'$ where M_i, M_i' are copies of the X-manifold M for $i = 1, 2, 3, 4$. Consider the X-cobordisms

$$W_0 = (C_3 \amalg C_4, M_3, M_3' \amalg \partial C_4) \quad \text{and} \quad W_1 = (C_1 \amalg C_2, -\partial C_1 \amalg M_2, M_2')$$

shown in Figure I.5. Clearly, $\tau(W_0) = \mathrm{id}_P \otimes \eta_M^-$ and $\tau(W_1) = \eta_M \otimes \mathrm{id}_P$.

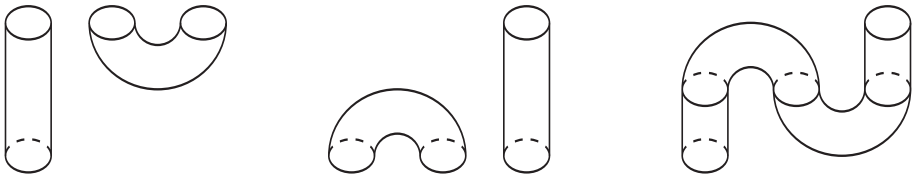


Figure I.5. The X-cobordisms W_0, W_1, and W.

Let (W, M_3, M_2') be the X-cobordism obtained by gluing W_1 on the top of W_0 along the identification homeomorphism

$$M_3' \amalg \partial C_4 = M \amalg (-M) \amalg M = -\partial C_1 \amalg M_2$$

of the top base of W_0 onto the bottom base of W_1; see Figure I.5. By Axiom (1.2.6), $\tau(W) = \tau(W_1)\,\tau(W_0)$. On the other hand, its is clear that W is X-homeomorphic to the cylinder X-cobordism C via an X-homeomorphism extending the identification maps of the bases $M_3 = M$, $M_2' = M$. Therefore

$$(\eta_M \otimes \mathrm{id}_P)(\mathrm{id}_P \otimes \eta_M^-) = \tau(W_1)\,\tau(W_0) = \tau(W) = \tau(C) = \mathrm{id}_P.$$

Replacing here M, P with $-M, Q$ respectively, and using the formulas $\eta_{-M} = \eta_M \sigma$ and $\eta_{-M}^- = \sigma \eta_M^-$, we obtain a formula equivalent to $(\mathrm{id}_Q \otimes \eta_M)(\eta_M^- \otimes \mathrm{id}_Q) = \mathrm{id}_Q$. Now, by Lemma 5.1 the pairing η_M is nondegenerate. □

5.3 Proof of Lemma 1.3.2. Consider the X-cobordisms

$$W_0 = (W, M_0, M_1 \amalg M) \amalg (-M \times [0,1], -M \times 0, -M \times 1)$$

and

$$W_1 = (M_1 \times [0,1], M_1 \times 0, M_1 \times 1) \amalg (M \times [0,1], (M \times 0) \amalg (-M \times 1), \emptyset)$$

shown in Figure I.6. Both cobordisms are endowed with maps to X induced by the characteristic map $g\colon W \to X$ and its restrictions to M_1 and M. Gluing W_1 on the top of W_0 along the identification homeomorphism

$$M_1 \amalg M \amalg (-M \times 1) = (M_1 \times 0) \amalg (M \times 0) \amalg (-M \times 1)$$

of the top base of W_0 on the bottom base of W_1 we obtain the X-cobordism $W' = (W, M_0 \amalg -M, M_1, g)$.

For any vector $u \in A_{M_0}$, we can expand

$$\tau(W)(u) = \sum_j u_j' \otimes u_j,$$

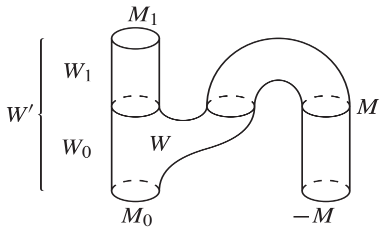


Figure I.6. The X-cobordisms W, W_0, W_1, and W'.

where j runs over a finite set of indices and $u'_j \in A_{M_1}, u_j \in A_M$. Then for any vector $v \in A_{-M}$,

$$\tau(W')(u \otimes v) = \tau(W_1)\,\tau(W_0)(u \otimes v) = \tau(W_1)\Big(\sum_j u'_j \otimes u_j \otimes v\Big) = \sum_j u'_j\,\eta_M(u_j, v).$$

The resulting identity $\tau(W')(u \otimes v) = \sum_j u'_j\,\eta_M(u_j, v)$ is equivalent to the claim of the lemma. □

5.4 Proof of Lemma 1.3.3. The X-cobordism (V, M_0, M_1) can be obtained by gluing three X-cobordisms

$$\begin{aligned} V_0 &= (M_0 \times [0,1], M_0 \times 0, M_0 \times 1) \amalg (-M'_0 \times [0,1], \emptyset, (M'_0 \times 0) \amalg (-M'_0 \times 1)),\\ V_1 &= (W, M_0 \amalg M'_0, M_1 \amalg M'_1) \amalg (-M'_0 \times [0,1], -M'_0 \times 0, -M'_0 \times 1),\\ V_2 &= (M_1 \times [0,1], M_1 \times 0, M_1 \times 1) \amalg (M'_0 \times [0,1], (M'_0 \times 0) \amalg (-M'_0 \times 1), \emptyset); \end{aligned}$$

see Figure I.7. All these cobordisms are endowed with maps to X induced by the characteristic map $W \to X$ and its restrictions to M_0, M_1, and M'_0. The gluing of V_1 on the top of V_0 is done along the obvious identification homeomorphism

$$(M_0 \times 1) \amalg (M'_0 \times 0) \amalg (-M'_0 \times 1) = M_0 \amalg M'_0 \amalg -M'_0 \times 0$$

of the top base of V_0 on the bottom base of V_1. The gluing of V_2 on the top of V_1 is done similarly along the X-homeomorphism

$$\mathrm{id}_{M_1} \amalg f \amalg \mathrm{id}_{-M'_0} \colon M_1 \amalg M'_1 \amalg -M'_0 \to M_1 \amalg M'_0 \amalg -M'_0$$

of the top base of V_1 on the bottom base of V_2.

Pick a finite expansion $\eta^-_{-M'_0}(1_K) = \sum_i p_i \otimes q_i$, where $p_i \in A_{M'_0}$ and $q_i \in A_{-M'_0}$.

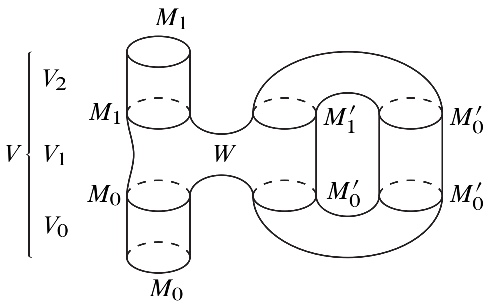


Figure I.7. The X-cobordisms V_0, V_1, V_2, and V.

For any $p \in A_{M_0}$ and $q \in (A_{M_1})^*$,

$$\begin{aligned}
q(\tau(V)(p)) &= q(\tau(V_2)\,\tau(V_1)\,\tau(V_0)(p)) \\
&= \sum_i q(\tau(V_2)\,\tau(V_1)(p \otimes p_i \otimes q_i)) \\
&= \sum_i q(\tau(V_2)(\tau(W)(p \otimes p_i) \otimes q_i)) \\
&= \sum_i q(\mathrm{id}_{A_{M_1}} \otimes \eta_{M_0'})(\mathrm{id}_{A_{M_1}} \otimes f_\# \otimes \mathrm{id}_{A_{-M_0'}})(\tau(W)(p \otimes p_i) \otimes q_i) \\
&= \sum_i \eta_{M_0'}(q \otimes f_\# \otimes \mathrm{id}_{A_{-M_0'}})(\tau(W)(p \otimes p_i) \otimes q_i) \\
&= \sum_i \eta_{M_0'}((q \otimes f_\#)\tau(W)(p \otimes p_i), q_i) \\
&= \mathrm{Tr}(A_{M_0'} \to A_{M_0'},\, x \mapsto (q \otimes f_\#)\tau(W)(p \otimes x)) \\
&= q(\mathrm{Tr}_{f_\#}(\tau(W))(p)).
\end{aligned}$$

The last two equalities follow respectively from (5.1.c) and the definition of the partial trace $\mathrm{Tr}_{f_\#}(\tau(W))$. Therefore $\tau(V) = \mathrm{Tr}_{f_\#}(\tau(W))$. □

Chapter II
Group-algebras

II.1 G-algebras

1.1 Basic definitions. Let G be a group. A *G-graded algebra* or, briefly, a *G-algebra* over the ring K is an associative algebra L over K endowed with a decomposition

$$L = \bigoplus_{\alpha \in G} L_\alpha$$

such that $L_\alpha L_\beta \subset L_{\alpha\beta}$ for any $\alpha, \beta \in G$ and L has a (right and left) unit $1_L \in L_1$ where 1 is the neutral element of G. We do not exclude the possibility that $1_L = 0$ which happens if and only if $L_\alpha = 0$ for all $\alpha \in G$. Then L is said to be *trivial*.

A *homomorphism of G-algebras* $L \to L'$ is an algebra homomorphism carrying 1_L to $1_{L'}$ and L_α to L'_α for all $\alpha \in G$.

The *direct sum* $L \oplus L'$ of G-algebras L, L' is a G-algebra defined by $(L \oplus L')_\alpha = L_\alpha \oplus L'_\alpha$ for all $\alpha \in G$ with coordinate-wise multiplication and unit $1_L + 1_{L'}$. The *tensor product* $L \otimes L'$ of G-algebras L, L' is a G-algebra defined by $(L \otimes L')_\alpha = L_\alpha \otimes L'_\alpha$ with multiplication induced by multiplication in L, L' in the obvious way and unit $1_L \otimes 1_{L'}$. The *dual* $\overline{L}$ of a G-algebra L is defined as the same module L with the same unit and opposite multiplication $a \circ b = ba$. The decomposition $\overline{L} = \bigoplus_{\alpha \in G} \overline{L}_\alpha$ is given by $\overline{L}_\alpha = L_{\alpha^{-1}}$.

A fundamental example of a G-algebra is the group algebra $L = K[G]$ with $L_\alpha = K\alpha$ for all $\alpha \in G$. In the next subsection, we generalize this G-algebra in two different directions. For more examples see Section VIII.5.

1.2 Examples. 1. Let $q \colon G' \to G$ be a group homomorphism. For $\alpha \in G$, let L_α be the K-submodule of the group algebra $K[G']$ generated over K by the set $q^{-1}(\alpha) \subset G'$. Clearly, $K[G'] = \bigoplus_\alpha L_\alpha$ is a G-algebra. For $G' = G$ and $q = \mathrm{id}$, we recover $K[G]$.

2. Let $\{\theta_{\alpha,\beta} \in K^*\}_{\alpha,\beta \in G}$ be a normalized 2-cocycle of G with values in the multiplicative group K^* of invertible elements of K. Thus, $\theta_{1,1} = 1$ and

$$\theta_{\alpha,\beta}\, \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta\gamma}\, \theta_{\beta,\gamma} \tag{1.2.a}$$

for any $\alpha, \beta, \gamma \in G$. We define a G-algebra $L = L^\theta$ as follows. For $\alpha \in G$, let L_α be the free K-module of rank one generated by a vector l_α, i.e., $L_\alpha = Kl_\alpha$. Multiplication is defined by $l_\alpha l_\beta = \theta_{\alpha,\beta} l_{\alpha\beta}$ and then extended to L by linearity. The associativity of multiplication follows from (1.2.a). Substituting $\beta = \gamma = 1$ in (1.2.a), we obtain that $\theta_{\alpha,1} = 1$ for all $\alpha \in G$. Substituting $\alpha = \beta = 1$ in (1.2.a), we obtain that $\theta_{1,\gamma} = 1$ for all $\gamma \in G$. Therefore $1_L = l_1 \in L_1$ is a unit of L.

It is easy to see that the isomorphism class of the G-algebra L^θ depends only on the cohomology class $\theta \in H^2(G; K^*)$ represented by the cocycle $\{\theta_{\alpha,\beta}\}_{\alpha,\beta\in G}$. For $\theta = 0 \in H^2(G; K^*)$, we recover $K[G]$.

II.2 Inner products and Frobenius G-algebras

2.1 Inner products on G-algebras. An *inner product* on a G-algebra L is a symmetric K-bilinear form $\eta\colon L \otimes L \to K$ such that

(2.1.1) $\eta(L_\alpha \otimes L_\beta) = 0$ if $\alpha\beta \neq 1$;

(2.1.2) the restriction of η to $L_\alpha \otimes L_{\alpha^{-1}}$ is nondegenerate for all $\alpha \in G$;

(2.1.3) $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in L$.

An inner product η on L is entirely determined by the linear functional $f_\eta \in L_1^* = \mathrm{Hom}_K(L_1, K)$ carrying $\ell \in L_1$ to $\eta(\ell, 1_L)$. Indeed, for $a \in L_\alpha, b \in L_\beta$, we have $\eta(a,b) = \eta(a, b1_L) = f_\eta(ab)$ if $\alpha\beta = 1$ and $\eta(a,b) = 0$ otherwise. This rule allows us to derive a bilinear form on L satisfying Axioms (2.1.1) and (2.1.2) from any vector in L_1^*. This bilinear form is an inner product if and only if it is symmetric and its restriction to $L_\alpha \otimes L_{\alpha^{-1}}$ is nondegenerate for all $\alpha \in G$.

2.2 Frobenius G-algebras. A *Frobenius G-algebra* is a pair (L, η) where $L = \bigoplus_{\alpha\in G} L_\alpha$ is a G-algebra such that each L_α is a projective K-module of finite type and η is an inner product on L. For example, $K[G]$ is a Frobenius G-algebra with inner product η defined by $\eta(\alpha, \beta) = 1$ if $\alpha\beta = 1$ and $\eta(\alpha,\beta) = 0$ if $\alpha\beta \neq 1$ for $\alpha, \beta \in G$.

The direct sum and the tensor product of Frobenius G-algebras L, L' are Frobenius: the inner products on L, L' extend to $L \oplus L'$ (resp. to $L \otimes L'$) by linearity (resp. by multiplicativity). The inner product on L induces an inner product on the dual G-algebra $\overline{L}$ via the equality of the underlying modules $\overline{L} = L$. This turns $\overline{L}$ into a Frobenius G-algebra.

For $G = \{1\}$, a Frobenius G-algebra is just a pair (an associative unital algebra L whose underlying K-module is projective of finite type, a non-degenerate symmetric bilinear form $\eta\colon L \otimes L \to K$ such that $\eta(ab, c) = \eta(a, bc)$ for all $a, b, c \in L$). For $G = \{1\}$, Frobenius G-algebras are briefly called *Frobenius algebras*.

2.3 Examples. 1. Consider the G-algebra $L = K[G']$ from Example 1.2.1 and suppose additionally that the kernel of the homomorphism $q\colon G' \to G$ is finite. For $a, b \in G'$, set $\eta(a,b) = 0$ if $ab \neq 1$ and $\eta(a,b) = 1$ if $ab = 1$. This extends by bilinearity to an inner product η on L. It is clear that (L, η) is a Frobenius G-algebra.

2. Consider the G-algebra $L = \bigoplus_{\alpha\in G} L_\alpha$ from Example 1.2.2. We define an inner product η on L by $\eta(l_\alpha, l_{\alpha^{-1}}) = \theta_{\alpha,\alpha^{-1}}$ for all α and $\eta(l_\alpha, l_\beta) = 0$ for $\beta \neq \alpha^{-1}$. Let us verify the axioms of an inner product. Substituting $\beta = \alpha^{-1}$ and $\gamma = \alpha$ in

(1.2.a) and using that $\theta_{1,\alpha} = \theta_{\alpha,1} = 1$, we obtain that $\theta_{\alpha,\alpha^{-1}} = \theta_{\alpha^{-1},\alpha}$ for all $\alpha \in G$. Hence η is symmetric. Axioms (2.1.1) and (2.1.2) are obvious. A direct computation shows that $\eta(l_\alpha, l_\beta) = \eta(l_\alpha l_\beta, 1_L)$ for all α, β. Therefore $\eta(a,b) = \eta(ab, 1_L)$ for all $a, b \in L$. This implies Axiom (2.1.3).

II.3 Crossed Frobenius G-algebras

3.1 Crossed G-algebras. When one attempts to introduce a commutativity constraint on a G-algebra L, the following difficulty arises. For $a \in L_\alpha$, $b \in L_\beta$, we have $ab \in L_{\alpha\beta}$ and $ba \in L_{\beta\alpha}$. Therefore a and b do not commute unless $\alpha\beta = \beta\alpha$ or $ab = ba = 0$. These conditions are too restrictive. We introduce a milder constraint $ba = a'b$ where $a' \in L_{\beta\alpha\beta^{-1}}$ is the image of a under a certain algebra automorphism of L determined by β. This leads us to a "crossed" structure on L. The exact definition of a crossed G-algebra below is motivated by the study of HQFTs in the next chapter.

In the sequel we use the following notation: for an element c of a K-algebra L, the K-linear homomorphism $L \to L$ carrying $a \in L$ to ca is denoted μ_c. The group of K-linear automorphisms of L is denoted $\mathrm{Aut}(L)$.

A G-algebra $L = \bigoplus_{\alpha\in G} L_\alpha$ is *crossed* if each L_α is a projective K-module of finite type and L is endowed with a group homomorphism $\varphi\colon G \to \mathrm{Aut}(L)$ so that

(3.1.1) for each $\beta \in G$, the homomorphism $\varphi_\beta = \varphi(\beta)\colon L \to L$ is an algebra automorphism of L carrying L_α to $L_{\beta\alpha\beta^{-1}}$ for all $\alpha \in G$;

(3.1.2) for any $a \in L$ and $b \in L_\beta$, we have $\varphi_\beta(a)b = ba$;

(3.1.3) $\varphi_\beta|_{L_\beta} = \mathrm{id}$, for all $\beta \in G$;

(3.1.4) for any $\alpha, \beta \in G$ and $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$,

$$\mathrm{Tr}(\mu_c\,\varphi_\beta\colon L_\alpha \to L_\alpha) = \mathrm{Tr}(\varphi_{\alpha^{-1}}\mu_c\colon L_\beta \to L_\beta). \tag{3.1.a}$$

Note a few corollaries of the definition. Axioms (3.1.2) and (3.1.3) imply that $ab = ba$ provided a and b belong to the same homogeneous component L_α of L. In particular, $L_1 \subset L$ is a commutative associative unital K-algebra. The group G acts on L_1 by the algebra automorphisms $\{\varphi_\beta\}_{\beta\in G}$. Axiom (3.1.2) with $\beta = 1$ implies that elements of L_1 commute with all elements of L. Axiom (3.1.4) with $\beta = 1$ and $c = 1_L \in L_1$ implies that for any $\alpha \in G$,

$$\mathrm{Dim}\, L_\alpha = \mathrm{Tr}(\mathrm{id}\colon L_\alpha \to L_\alpha) = \mathrm{Tr}(\varphi_{\alpha^{-1}}\colon L_1 \to L_1).$$

In particular, if K is a field of characteristic zero, then the dimensions of all $\{L_\alpha\}$ are determined by the character of the representation $\varphi|_{L_1}\colon G \to \mathrm{Aut}(L_1)$.

3.2 Crossed Frobenius G-algebras. We now give the main definition of this chapter. A *crossed Frobenius G-algebra* is a crossed G-algebra with inner product η invariant under the action of G. Note that η is G-invariant if and only if the linear functional f_η defined in Section 2.1 is G-invariant.

The group algebra $K[G]$ with action of G by conjugations and the inner product defined in Section 2.2 is a crossed Frobenius G-algebra. This example will be generalized in two different directions in the next subsection.

The direct sum and the tensor product of crossed Frobenius G-algebras L, L' are crossed Frobenius G-algebras: the actions of G on L and L' extend to $L \oplus L'$ (resp. to $L \otimes L'$) by linearity (resp. by multiplicativity). The action of G on L induces an action of G on the dual G-algebra $\overline{L}$ via the equality of the underlying modules $\overline{L} = L$; this turns $\overline{L}$ into a crossed Frobenius G-algebra. Another useful operation on a crossed Frobenius G-algebra (L, η, φ) consists in rescaling the inner product: for $k \in K^*$, the triple $(L, k\eta, \varphi)$ is also a crossed Frobenius G-algebra.

We define a category $\mathcal{Q}(G) = \mathcal{Q}(G; K)$ whose objects are crossed Frobenius G-algebras over K. A morphism in this category is an isomorphism of G-algebras preserving the inner product and commuting with the action of G.

For $G = \{1\}$, crossed Frobenius G-algebras are nothing but commutative Frobenius algebras over K.

3.3 Examples. 1. Consider the Frobenius G-algebra $L = K[G']$ derived from a group homomorphism $q : G' \to G$ with finite kernel $\Gamma = \operatorname{Ker} q$ (see Examples 1.2.1 and 2.3.1). Suppose that $q(G') = G$ and that Γ lies in the center of G' (in particular, Γ is abelian). For $\beta \in G$, define $\varphi_\beta : L \to L$ by $\varphi_\beta(a) = bab^{-1}$, where b is an arbitrary element of $q^{-1}(\beta)$ and a runs over G'. We claim that L is a crossed Frobenius G-algebra. Axioms (3.1.1)–(3.1.3) and the invariance of the inner product under the action of G are straightforward. We check Axiom (3.1.4). Let $\alpha, \beta \in G$. Note that for $a \in q^{-1}(\alpha), b \in q^{-1}(\beta)$, the commutator $aba^{-1}b^{-1}$ depends only on α and β. Denote this commutator $w_{\alpha,\beta}$. To check (3.1.a), it suffices to consider the case $c \in q^{-1}(\alpha\beta\alpha^{-1}\beta^{-1}) \subset L_{\alpha\beta\alpha^{-1}\beta^{-1}}$. Pick any $b_0 \in q^{-1}(\beta)$. Then

$$\begin{aligned} \operatorname{Tr}(\mu_c \varphi_\beta : L_\alpha \to L_\alpha) &= \operatorname{card}\{a \in q^{-1}(\alpha) \mid c b_0 a b_0^{-1} = a\} \\ &= \begin{cases} \operatorname{card} q^{-1}(\alpha) & \text{if } c = w_{\alpha,\beta}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\operatorname{card} q^{-1}(\alpha) = |\Gamma|$, where $|\Gamma|$ is the order of Γ. Similarly,

$$\operatorname{Tr}(\varphi_{\alpha^{-1}} \mu_c : L_\beta \to L_\beta) = \begin{cases} |\Gamma| & \text{if } c = w_{\alpha,\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

This proves (3.1.a).

2. Consider the G-algebra $L = \bigoplus_{\alpha \in G} K\alpha$ derived from a normalized 2-cocycle $\{\theta_{\alpha,\beta} \in K^*\}_{\alpha,\beta \in G}$; see Example 1.2.2. We provide L with inner product η as in

Section 2.3.2. Note that for $\alpha, \beta \in G$, multiplication in L induces an isomorphism $L_\alpha \otimes L_\beta \to L_{\alpha\beta}$ of free K-modules of rank one. This implies that there is a unique $\varphi_\beta(l_\alpha) \in L_{\beta\alpha\beta^{-1}}$ such that $\varphi_\beta(l_\alpha)l_\beta = l_\beta l_\alpha$. The assignment $l_\alpha \mapsto \varphi_\beta(l_\alpha)$ extends by linearity to a K-linear automorphism φ_β of L. We claim that (L, η, φ) is a crossed Frobenius G-algebra. It will be denoted L^θ.

We first verify the multiplicativity of φ. Pick $\beta, \beta', \alpha \in G$. Then

$$\begin{aligned}\varphi_{\beta'\beta}(l_\alpha)l_{\beta'\beta} &= l_{\beta'\beta}l_\alpha = \theta^{-1}_{\beta',\beta} l_{\beta'} l_\beta l_\alpha \\ &= \theta^{-1}_{\beta',\beta} l_{\beta'} \varphi_\beta(l_\alpha) l_\beta = \theta^{-1}_{\beta',\beta} \varphi_{\beta'}\varphi_\beta(l_\alpha) l_{\beta'} l_\beta = \varphi_{\beta'}\varphi_\beta(l_\alpha) l_{\beta'\beta}.\end{aligned}$$

Therefore $\varphi_{\beta'\beta}(l_\alpha) = \varphi_{\beta'}\varphi_\beta(l_\alpha)$. Hence, $\varphi_{\beta'\beta} = \varphi_{\beta'}\varphi_\beta$.

Let us check Axiom (3.1.1). For $\alpha, \alpha', \beta \in G$,

$$\begin{aligned}\varphi_\beta(l_\alpha l_{\alpha'})l_\beta &= \theta_{\alpha,\alpha'}\varphi_\beta(l_{\alpha\alpha'})l_\beta = \theta_{\alpha,\alpha'} l_\beta l_{\alpha\alpha'} = l_\beta l_\alpha l_{\alpha'} \\ &= \varphi_\beta(l_\alpha) l_\beta l_{\alpha'} = \varphi_\beta(l_\alpha)\varphi_\beta(l_{\alpha'}) l_\beta.\end{aligned}$$

Therefore $\varphi_\beta(l_\alpha l_{\alpha'}) = \varphi_\beta(l_\alpha)\varphi_\beta(l_{\alpha'})$. Substituting $\alpha = \alpha' = 1$, we obtain that $\varphi_\beta(1_L) = 1_L$. Hence $\varphi_\beta : L \to L$ ia an algebra homomorphism.

Axioms (3.1.2) and (3.1.3) follow directly from the definition of φ_β. Let us check Axiom (3.1.4). The homomorphism $\mu_c\,\varphi_\beta : L_\alpha \to L_\alpha$ carries l_α into kl_α for some $k \in K$. The homomorphism $\varphi_{\alpha^{-1}}\mu_c : L_\beta \to L_\beta$ carries l_β into $k'l_\beta$ for some $k' \in K$. Then

$$kl_\alpha l_\beta = \mu_c\,\varphi_\beta(l_\alpha)l_\beta = c\,l_\beta l_\alpha = l_\alpha \varphi_{\alpha^{-1}}(c\,l_\beta) = k' l_\alpha l_\beta.$$

Therefore $k = k'$ and

$$\mathrm{Tr}(\mu_c\,\varphi_\beta : L_\alpha \to L_\alpha) = k = k' = \mathrm{Tr}(\varphi_{\alpha^{-1}}\mu_c : L_\beta \to L_\beta).$$

It remains to verify the invariance of η under the action of G. For $a, b \in L$,

$$\eta(\varphi_\beta(a), \varphi_\beta(b)) = \eta(\varphi_\beta(a)\varphi_\beta(b), 1_L) = \eta(\varphi_\beta(ab), 1_L).$$

Note that $\bigoplus_{\alpha \neq 1} L_\alpha$ is orthogonal to L_1 and $\varphi_\beta|_{L_1} = \mathrm{id}$. Therefore

$$\eta(\varphi_\beta(ab), 1_L) = \eta(ab, 1_L) = \eta(a, b).$$

The following explicit expression for $\varphi_\beta(l_\alpha)$ follows from the definition of multiplication in L and the equality $\varphi_\beta(l_\alpha)l_\beta = l_\beta l_\alpha$:

$$\varphi_\beta(l_\alpha) = \theta_{\beta,\alpha}\,\theta^{-1}_{\beta\alpha\beta^{-1},\beta}\,l_{\beta\alpha\beta^{-1}}.$$

We can compute $\theta_{\beta\alpha\beta^{-1},\beta}$ from the equalities

$$\theta_{\beta\alpha,\beta^{-1}}\,\theta_{\beta\alpha\beta^{-1},\beta} = \theta_{\beta^{-1},\beta}\,\theta_{\beta\alpha,1} = \theta_{\beta^{-1},\beta}.$$

This gives

$$\varphi_\beta(l_\alpha) = \theta^{-1}_{\beta^{-1},\beta}\,\theta_{\beta,\alpha}\,\theta_{\beta\alpha,\beta^{-1}}\,l_{\beta\alpha\beta^{-1}}. \tag{3.3.a}$$

It is easy to see that the isomorphism class of the crossed Frobenius G-algebra L^θ depends only on the cohomology class $\theta \in H^2(G;K^*)$ represented by the cocycle $\{\theta_{\alpha,\beta}\}_{\alpha,\beta}$. It is obvious that $L^{\theta+\theta'} = L^\theta \otimes L^{\theta'}$ for any $\theta, \theta' \in H^2(G;K^*)$. For $\theta = 0$, we have $L^\theta = K[G]$.

3.4 Pull-backs. We define one more transformation of group-algebras, the *pull-back* along a group homomorphism $q\colon G' \to G$. The pull-back of a G-algebra L along q is the G'-algebra $L' = q^*(L)$ defined by $L'_a = L_{q(a)}$ for all $a \in G'$. Multiplication in L' is induced by multiplication in L in the obvious way. An inner product η on L induces an inner product η' on L' as follows: for any $a, b \in G'$ and $x \in L'_a = L_{q(a)}$, $y \in L'_b = L_{q(b)}$, set $\eta'(x,y) = \eta(x,y)$ if $ab = 1$ and $\eta'(x,y) = 0$ otherwise. Similarly, a crossed structure φ on L induces a crossed structure φ' on L' by

$$\varphi'_b = \varphi_{q(b)}\colon L'_a = L_{q(a)} \to L_{q(bab^{-1})} = L'_{bab^{-1}}$$

for $a, b \in G'$. Combining these constructions, we obtain a pull-back $L \mapsto q^*(L)$ transforming a crossed Frobenius G-algebra into a crossed Frobenius G'-algebra. In the case where G' is a subgroup of G and $q\colon G' \to G$ is the inclusion, we call $q^*(L)$ the *restriction* of L to G'.

II.4 Transfer

Fix a subgroup H of G of finite index. We show that any (crossed Frobenius) H-algebra gives rise to a (crossed Frobenius) G-algebra, called its transfer.

4.1 Transfer of H-algebras. Let $L = \bigoplus_{\alpha\in H} L_\alpha$ be an H-algebra over K. Its *transfer* to G is a G-algebra $\widetilde{L} = \bigoplus_{\alpha\in G}\widetilde{L}_\alpha$ over K defined as follows. Let $H\backslash G$ be the set of right cosets of H in G. For each $i \in H\backslash G$, fix a representative $\omega_i \in G$ of i so that $i = H\omega_i$. (It is convenient but not necessary to take $\omega_i = 1$ for $i = H$.) For $\alpha \in G$, set

$$N(\alpha) = \{i \in H\backslash G \mid \omega_i\,\alpha\,\omega_i^{-1} \in H\}$$

and

$$\widetilde{L}_\alpha = \bigoplus_{i\in N(\alpha)} L_{\omega_i\alpha\omega_i^{-1}}.$$

In particular, if α is not conjugate to elements of H, then $\widetilde{L}_\alpha = 0$.

We provide $\widetilde{L} = \bigoplus_\alpha \widetilde{L}_\alpha$ with coordinate-wise multiplication. Thus, for $\alpha, \beta \in G$, the multiplication $\widetilde{L}_\alpha \times \widetilde{L}_\beta \to \widetilde{L}_{\alpha\beta}$ restricted to $L_{\omega_i \alpha \omega_i^{-1}} \times L_{\omega_j \beta \omega_j^{-1}}$ is 0, if $i \neq j$, and is induced by multiplication in L

$$L_{\omega_i \alpha \omega_i^{-1}} \times L_{\omega_i \beta \omega_i^{-1}} \to L_{\omega_i \alpha\beta \omega_i^{-1}}$$

if $i = j \in N(\alpha) \cap N(\beta)$. Clearly, $\widetilde{L}$ is an associative algebra.

By definition, the algebra $\widetilde{L}_1$ is a direct sum of $[G\colon H]$ copies of L_1 labeled by elements of $N(1) = H\backslash G$. The corresponding sum of copies of $1_L \in L_1$ is the unit $1_{\widetilde{L}} \in \widetilde{L}_1$ of $\widetilde{L}$. Thus $\widetilde{L}$ is a G-algebra. It is called the *transfer* of L.

An inner product η on L induces a symmetric bilinear form $\tilde{\eta}\colon \widetilde{L} \otimes \widetilde{L} \to K$ whose restriction to $\widetilde{L}_\alpha \otimes \widetilde{L}_\beta$ is 0 if $\alpha\beta \neq 1$ and is given by

$$\tilde{\eta}|_{\widetilde{L}_\alpha \otimes \widetilde{L}_{\alpha^{-1}}} = \bigoplus_{i \in N(\alpha) = N(\alpha^{-1})} \eta|_{L_{\omega_i \alpha \omega_i^{-1}} \otimes L_{\omega_i \alpha^{-1} \omega_i^{-1}}}$$

if $\beta = \alpha^{-1}$. Clearly, $\tilde{\eta}$ is an inner product on $\widetilde{L}$. Thus, the transfer of a Frobenius H-algebra is a Frobenius G-algebra.

4.2 Transfer for crossed H-algebras. Let (L, φ) be a crossed H-algebra. We provide the G-algebra $\widetilde{L}$ with a crossed structure $\widetilde{\varphi}$ as follows. The group G acts on $H\backslash G$ by $\beta(i) = i\beta^{-1}$, where $\beta \in G$ and $i \in H\backslash G$. Then

$$H\omega_{\beta(i)} = \beta(i) = i\beta^{-1} = H\omega_i\beta^{-1}.$$

Set $\beta_i = \omega_{\beta(i)}\beta\omega_i^{-1}$. The previous equalities show that $\beta_i \in H$.

The fixed point set of the bijection $H\backslash G \to H\backslash G$, $i \mapsto \beta(i)$ is $N(\beta)$. Therefore, for an arbitrary $\alpha \in G$, this bijection carries $N(\alpha)$ bijectively onto $N(\beta\alpha\beta^{-1})$. For each $i \in N(\alpha)$, consider the K-linear isomorphism

$$\varphi_{\beta_i} : L_{\omega_i \alpha \omega_i^{-1}} \to L_{\beta_i \omega_i \alpha \omega_i^{-1} \beta_i^{-1}} = L_{\omega_{\beta(i)} \beta\alpha\beta^{-1} \omega_{\beta(i)}^{-1}}. \tag{4.2.a}$$

We define $\widetilde{\varphi}_\beta\colon \widetilde{L}_\alpha \to \widetilde{L}_{\beta\alpha\beta^{-1}}$ to be the direct sum of these isomorphisms over all $i \in N(\alpha)$. This extends by additivity to a K-linear automorphism $\widetilde{\varphi}_\beta$ of $\widetilde{L}$.

4.2.1 Lemma. *The pair $(\widetilde{L}, \widetilde{\varphi})$ is a crossed G-algebra.*

Proof. A direct computation shows that $(\beta\beta')_i = \beta_{\beta'(i)}\beta'_i$ for any $\beta, \beta' \in G$ and $i \in H\backslash G$. This implies that $\widetilde{\varphi}_{\beta\beta'} = \widetilde{\varphi}_\beta\, \widetilde{\varphi}_{\beta'}$. It follows from the definitions that $\widetilde{\varphi}_\beta$ is an algebra automorphism of $\widetilde{L}$ and $\widetilde{\varphi}_1 = \mathrm{id}$. Hence $\widetilde{\varphi}$ is an action of G on $\widetilde{L}$ by algebra automorphisms.

We must verify Axioms (3.1.1)–(3.1.4) for $\widetilde{\varphi}$. Axiom (3.1.1) is clear. For $i \in N(\beta)$, we have $\beta(i) = i$ and $\beta_i = \omega_i\beta\omega_i^{-1}$. By Axiom (3.1.3) for L, the homomorphism (4.2.a) with $\alpha = \beta$ is the identity. This implies Axiom (3.1.3) for $\widetilde{\varphi}$.

Let us verify Axiom (3.1.2). Let $a \in L_{\omega_i \alpha \omega_i^{-1}} \subset \widetilde{L}_\alpha$ and $b \in L_{\omega_j \beta \omega_j^{-1}} \subset \widetilde{L}_\beta$, where $i \in N(\alpha)$ and $j \in N(\beta)$. By definition,

$$\widetilde{\varphi}_\beta(a) = \varphi_{\beta_i}(a) \in L_{\omega_{\beta(i)} \beta\alpha\beta^{-1} \omega_{\beta(i)}^{-1}}.$$

If $i \neq j$, then

$$\beta(i) \neq \beta(j) = j \quad \text{and} \quad ba = 0 = \widetilde{\varphi}_\beta(a) b.$$

If $i = j$, then $\beta(i) = \beta(j) = j$, $\beta_i = \omega_i \beta \omega_i^{-1}$, and by Axiom (3.1.2) for L,

$$\widetilde{\varphi}_\beta(a) b = \varphi_{\beta_i}(a) b = \varphi_{\omega_i \beta \omega_i^{-1}}(a) b = \varphi_{\omega_j \beta \omega_j^{-1}}(a) b = ba.$$

To verify Axiom (3.1.4), we must prove that for any $\alpha, \beta \in G$ and $c \in \widetilde{L}_{\alpha\beta\alpha^{-1}\beta^{-1}}$,

$$\operatorname{Tr}(\mu_c \, \widetilde{\varphi}_\beta : \widetilde{L}_\alpha \to \widetilde{L}_\alpha) = \operatorname{Tr}(\widetilde{\varphi}_{\alpha^{-1}} \mu_c : \widetilde{L}_\beta \to \widetilde{L}_\beta). \tag{4.2.b}$$

Since both sides are additive functions of c, it suffices to treat the case where

$$c \in L_{\omega_i \alpha\beta\alpha^{-1}\beta^{-1} \omega_i^{-1}} \subset \widetilde{L}_{\alpha\beta\alpha^{-1}\beta^{-1}}$$

where $i \in N(\alpha\beta\alpha^{-1}\beta^{-1})$. A direct application of the definitions shows that both sides of (4.2.b) are equal to zero unless $i \in N(\alpha) \cap N(\beta)$. If $i \in N(\alpha) \cap N(\beta)$, then the left-hand side of (4.2.b) is equal to the trace of the endomorphism $\mu_c \, \varphi_{\omega_i \beta \omega_i^{-1}}$ of $\widetilde{L}_{\omega_i \alpha \omega_i^{-1}}$ and the right-hand side of (4.2.b) is equal to the trace of the endomorphism $\varphi_{\omega_i \alpha^{-1} \omega_i^{-1}} \mu_c$ of $\widetilde{L}_{\omega_i \beta \omega_i^{-1}}$. The equality of these traces follows from Axiom (3.1.4) for L. □

4.3 Transfer of crossed Frobenius H-algebras. If (L, η, φ) is a crossed Frobenius H-algebra, then the constructions above give a crossed G-algebra $(\widetilde{L}, \widetilde{\varphi})$ with inner product $\tilde{\eta}$. The homomorphism (4.2.a) preserves η and therefore $\widetilde{\varphi}$ preserves $\tilde{\eta}$. Hence $(\widetilde{L}, \tilde{\eta}, \widetilde{\varphi})$ is a crossed Frobenius G-algebra.

As an exercise, the reader may verify that the isomorphism class of $(\widetilde{L}, \tilde{\eta}, \widetilde{\varphi})$ does not depend on the choice of the representatives $\{\omega_i\}_i$.

4.4 Example. As we know, any cohomology class $\theta \in H^2(H; K^*)$ determines a crossed Frobenius H-algebra $L^\theta = (L, \eta, \varphi)$. Its transfer $(\widetilde{L}, \tilde{\eta}, \widetilde{\varphi})$ to G is a crossed Frobenius G-algebra denoted $L^{G,H,\theta}$. By definition, $\widetilde{L}_1$ is a direct sum of $[G : H]$ copies of $L_1 = K$ labeled by elements of $H \backslash G$. The (left) action $\widetilde{\varphi}$ of G on $\widetilde{L}_1$ permutes these copies of K via the canonical left action of G on $H \backslash G$ used in Section 4.2. For the unit element i of a summand $L_1 = K$ of $\widetilde{L}$, we have $\tilde{\eta}(i, i) = \eta(l_1, l_1) = 1$. Given $k \in K^*$, the triple $(\widetilde{L}, k\tilde{\eta}, \widetilde{\varphi})$ is a crossed Frobenius G-algebra denoted $L^{G,H,\theta,k}$. In particular, $L^{G,H,\theta,1} = L^{G,H,\theta}$.

II.5 Semisimple crossed G-algebras

5.1 Non-singular and semisimple G-algebras. A G-algebra $L = \bigoplus_{\alpha \in G} L_\alpha$ over K is *nonsingular* if for any $\alpha \in G$ and any nonzero $a \in L_\alpha$, there is $b \in L_{\alpha^{-1}}$ such that $ab \neq 0$. Any Frobenius G-algebra (L, η) is nonsingular. Indeed, for a nonzero $a \in L_\alpha$, there is $b \in L_{\alpha^{-1}}$ such that $\eta(a, b) \neq 0$. Since $\eta(a, b) = \eta(ab, 1_L)$, we have $ab \neq 0$.

We say that a crossed G-algebra $L = \bigoplus_{\alpha \in G} L_\alpha$ is *semisimple* if L_1 is a direct sum of several (mutually annihilating) copies of K. For instance, the G-algebra L^θ associated with $\theta \in H^2(G; K^*)$ is semisimple since $L_1^\theta = K$. The direct sums, tensor products, pull-backs, duals, and transfers of semisimple crossed group-algebras are semisimple.

The aim of this section is to classify nonsingular semisimple crossed G-algebras over a field of characteristic zero. In the next section we shall classify semisimple Frobenius crossed G-algebras over a field of characteristic zero. Both classifications proceed in terms of finite G-sets and their 2-dimensional equivariant cohomology with coefficients in K^*. Here and below by a G-*set*, we mean a set with left action of G. We recall now the definition of equivariant cohomology of a G-set.

5.2 Equivariant cohomology. The G-equivariant cohomology of a G-set I with coefficients in K^* is defined by

$$H_G^*(I; K^*) = H^*(E \times_G I; K^*),$$

where E is the universal cover of $K(G, 1)$ with action of G by deck transformations and $E \times_G I$ is the quotient of $E \times I$ by the diagonal action of G (see [Bro], Section VII.7). Note that if I is a disjoint union of two G-sets I_1 and I_2, then

$$H_G^*(I; K^*) = H_G^*(I_1; K^*) \oplus H_G^*(I_2; K^*).$$

The equivariant cohomology of a G-set I can be computed in terms of stabilizers of elements of I. Denote the stabilizer $\{\alpha \in G \mid \alpha i = i\}$ of $i \in I$ by G_i. The conjugation by $\gamma \in G$ induces a group isomorphism $G_i \to G_{\gamma i} = \gamma G_i \gamma^{-1}$. Denote the induced isomorphism $H^*(G_i; K^*) \to H^*(G_{\gamma i}; K^*)$ by γ_*. If the G-set I is *transitive* in the sense that the action of G on I is transitive, then $E \times_G I$ is connected and the projection $E \times_G I \to E/G = K(G, 1)$ is a covering corresponding to G_i. Thus, $E \times_G I = K(G_i, 1)$ and $H_G^*(I; K^*) = H^*(G_i; K^*)$ for any $i \in I$. For $\gamma \in G$, the composition of the isomorphisms

$$H^*(G_i; K^*) = H_G^*(I; K^*) = H^*(G_{\gamma i}; K^*)$$

is γ_*. Since any G-set I is a disjoint union of transitive G-sets, we can describe an element of $H_G^*(I; K^*)$ as a function assigning to every $i \in I$ a cohomology class $\nabla_i \in H^*(G_i; K^*)$ such that $\nabla_{\gamma i} = \gamma_*(\nabla_i)$ for all $i \in I$ and $\gamma \in G$.

5.3 Basic idempotents. A fundamental isomorphism invariant of a semisimple crossed G-algebra $(L = \bigoplus_{\alpha\in G} L_\alpha, \varphi)$ is a G-set defined as follows. Consider a decomposition of L_1 as a direct sum of a finite number of copies of K, say $\{K_u\}_u$. Let $i_u \in K_u$ be the unit element of K_u. We have $1_L = \sum_u i_u$ and $i_u i_v = \delta_u^v i_u$ for any u, v, where δ is the Kronecker symbol. Each $r \in L_1$ can be uniquely expanded in the form $r = \sum_u r_u i_u$ with $r_u \in K$. It is clear that r is an idempotent (i.e., $r^2 = r$) if and only if $r_u \in \{0,1\}$ for all u. We call the set $\{u \mid r_u = 1\}$ the *support* of r. Now, the product of two idempotents is zero if and only if their supports are disjoint. This implies that the set $\{i_u\}_u$ is the unique basis of L_1 consisting of mutually annihilating idempotents. We denote the set $\{i_u\}_u \subset L_1$ by $\mathrm{Bas}(L)$ or, shorter, by I. The elements i_u of I are called the *basic idempotents* of L. Clearly, algebra automorphisms of L_1 preserve I set-wise. In particular, the action φ of G on L preserves I set-wise and turns I into a G-set.

Note that all elements of L_1 and, in particular, all basic idempotents lie in the center of L. The equalities $ii' = \delta_{i'}^{i} i$ for $i, i' \in I$ and $1_L = \sum_{i\in I} i$ imply that L is a direct sum of mutually annihilating subalgebras $\{iL\}_{i\in I}$:

$$L = \bigoplus_{i\in \mathrm{Bas}(L)} iL, \quad \text{where } iL = Li = \bigoplus_{\alpha\in G} iL_\alpha. \tag{5.3.a}$$

The subalgebras $\{iL\}_{i\in I}$ of L are permuted by the action of G on L.

5.4 The cohomology class ∇. The second fundamental invariant of a semisimple crossed G-algebra (L, φ) over K is an equivariant 2-dimensional cohomology class ∇ of the G-set $\mathrm{Bas}(L)$. It is defined under the assumption that K is a field of characteristic zero and L is nonsingular.

Fix $i \in I = \mathrm{Bas}(L)$. We first compute the dimension of $iL_\alpha \subset L_\alpha$ for all $\alpha \in G$. Applying (3.1.a) to $\beta = 1$ and $c = i \in L_1$ and using the assumptions that i is an idempotent and $\mathrm{char}\, K = 0$, we obtain that

$$\dim(iL_\alpha) = \mathrm{Tr}(a \mapsto ia : L_\alpha \to L_\alpha) = \mathrm{Tr}(a \mapsto \varphi_{\alpha^{-1}}(ia) : L_1 \to L_1).$$

The endomorphism $a \mapsto \varphi_{\alpha^{-1}}(ia)$ of L_1 can be computed on the basis I of L_1. It carries i to $\varphi_{\alpha^{-1}}(i) \in I$ and carries all elements of $I - \{i\}$ to 0. Hence,

$$\dim(iL_\alpha) = \begin{cases} 1 & \text{if } \varphi_\alpha(i) = i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.a}$$

In other words, $iL_\alpha \cong K$ for $\alpha \in G_i = \{\alpha \in G \mid \varphi_\alpha(i) = i\}$ and $iL_\alpha = 0$ for $\alpha \in G - G_i$. Pick a non-zero vector $s_\alpha \in iL_\alpha$ for every $\alpha \in G_i - \{1\}$ and set $s_1 = i \in iL_1$. For any $\alpha, \beta \in G_i$, we have $s_\alpha s_\beta = \nabla_{\alpha,\beta}\, s_{\alpha\beta}$ with $\nabla_{\alpha,\beta} \in K$. We claim that $\nabla_{\alpha,\beta} \in K^* = K - \{0\}$. Indeed, since L is nonsingular, $s_\beta L_{\beta^{-1}} \neq 0$. The equality $s_\beta i' L = 0$ for all $i' \in I - \{i\}$ implies that $s_\beta\, iL_{\beta^{-1}} \neq 0$ and so $s_\beta s_{\beta^{-1}} \neq 0$.

Then $s_\beta s_{\beta^{-1}} = ks_1 = ki$ with $k \in K^*$ and

$$(s_\alpha s_\beta)s_{\beta^{-1}} = s_\alpha(s_\beta s_{\beta^{-1}}) = ks_\alpha \neq 0. \tag{5.4.b}$$

Therefore $s_\alpha s_\beta \neq 0$ and $\nabla_{\alpha,\beta} \neq 0$.

The associativity of multiplication in L and the choice $s_1 = i$ imply that $\{\nabla_{\alpha,\beta}\}_{\alpha,\beta}$ is a normalized K^*-valued 2-cocycle of G_i. Let $\nabla_i \in H^2(G_i; K^*)$ be the cohomology class of this cocycle. Under a different choice of $\{s_\alpha\}_\alpha$ we obtain a cohomological 2-cocycle and the same ∇_i. It is easy to check that the mapping $i \mapsto \nabla_i$ satisfies the conditions formulated at the end of Section 5.2 and defines an element $\nabla = \nabla(L)$ of $H^2_G(I; K^*)$.

It is clear that if L is a direct sum of two nonsingular semisimple crossed G-algebras L^1, L^2, then

$$\mathrm{Bas}(L) = \mathrm{Bas}(L^1) \amalg \mathrm{Bas}(L^2) \quad \text{and} \quad \nabla(L) = \nabla(L^1) + \nabla(L^2). \tag{5.4.c}$$

5.5 Theorem. *Let K be a field of characteristic zero. The formula*

$$L \mapsto (\mathrm{Bas}(L), \nabla(L))$$

establishes a bijective correspondence between the isomorphism classes of nonsingular semisimple crossed G-algebras L over K and the isomorphism classes of pairs (a finite G-set I, an element of $H^2_G(I; K^*)$).

Here two pairs (a finite G-set, its equivariant cohomology class) are said to be *isomorphic* if there is a G-equivariant bijection between the underlying G-sets such that the induced isomorphism in cohomology carries one cohomology class into the other cohomology class.

Formulas (5.3.a) and (5.4.a) allow us to compute the dimension of L_α directly from the action of G on $I = \mathrm{Bas}(L)$: for all $\alpha \in G$,

$$\dim L_\alpha = \mathrm{card}\{i \in I \mid \alpha i = i\}.$$

In particular, $\dim L_1 = \mathrm{card}\, I$ as it should be because I is a basis of L_1. Clearly, $L_\alpha = 0$ if and only if the action of α on I has no fixed points.

Theorem 5.5 is proven in Section 5.7 The proof is based on a reduction to simple crossed G-algebras introduced in the next subsection.

5.6 Simple crossed G-algebras. A crossed G-algebra $L = \bigoplus_{\alpha\in G} L_\alpha$ is *simple* if it is semisimple and the action of G on L_1 is transitive on the set $I = \mathrm{Bas}(L)$. Observe that every semisimple crossed G-algebra L splits uniquely as a direct sum of mutually annihilating simple crossed G-algebras. Indeed, the algebra L_1 splits as a direct sum of its subalgebras $\{mL_1\}_m$ generated by the orbits $m \subset I$ of the action of G on I. Then L is a direct sum of the (mutually annihilating) subalgebras $mL = \bigoplus_{i\in m} iL$. Each mL is preserved by the action of G. Therefore mL is a simple crossed G-algebra.

Axioms (3.1.1)–(3.1.4) for mL directly follow from the corresponding axioms for L. Clearly, L is nonsingular if and only if mL is nonsingular for all m. In this way the study of nonsingular semisimple crossed G-algebras reduces to a study of nonsingular simple crossed G-algebras.

We now classify nonsingular simple crossed G-algebras over a field of characteristic zero K. Denote by $\mathrm{Sim}(G)$ the set of isomorphism classes of pairs (a nonsingular simple crossed G-algebra (L,φ) over K, a basic idempotent $i \in \mathrm{Bas}(L)$). The group G acts on $\mathrm{Sim}(G)$ by the formula $\gamma(L,\varphi,i) = (L,\varphi,\varphi_\gamma(i))$, where $\gamma \in G$. Denote by $\mathrm{Coh}(G)$ the set of pairs (a subgroup $H \subset G$ of finite index, a cohomology class $\theta \in H^2(H;K^*)$). The group G acts on $\mathrm{Coh}(G)$ as follows: $\gamma(H,\theta) = (\gamma H\gamma^{-1}, \gamma_*(\theta))$, where $\gamma \in G$ and $\gamma_*\colon H^2(H;K^*) \to H^2(\gamma H\gamma^{-1};K^*)$ is the isomorphism induced by the conjugation by γ.

5.6.1 Lemma. *Let K be a field of characteristic zero. Assigning to each pair (a nonsingular simple crossed G-algebra (L,φ) over K, a basic idempotent $i \in \mathrm{Bas}(L)$) the stabilizer $G_i \subset G$ of i and the cohomology class $\nabla_i \in H^2(G_i;K^*)$ defined in Section 5.4, we obtain a G-equivariant bijection* $\mathrm{Sim}(G) \to \mathrm{Coh}(G)$.

Proof. That the mapping $\mathrm{Sim}(G) \to \mathrm{Coh}(G)$ is G-equivariant follows directly from the definitions. We need only to prove that it is bijective.

As we know, any subgroup of finite index $H \subset G$ and any $\theta \in H^2(H;K^*)$ determine a crossed Frobenius G-algebra $\widetilde{L} = L^{G,H,\theta}$; see Section 4.4. It is simple because $\widetilde{L}_1$ is a direct sum of $[G\colon H]$ copies of $L_1^\theta = K$ labeled by elements of $H\backslash G$ and the action of G on $\widetilde{L}_1$ permutes these copies of K transitively. Since $\widetilde{L}$ is Frobenius, it is nonsingular. Let $i_\theta \in \mathrm{Bas}(\widetilde{L})$ be the unit element of the copy of $L_1^\theta = K$ corresponding to $H\backslash H \in H\backslash G$. The formula $(H,\theta) \mapsto (L^{G,H,\theta}, i_\theta)$ defines a mapping $\mathrm{Coh}(G) \to \mathrm{Sim}(G)$.

We claim that the mappings $\mathrm{Coh}(G) \to \mathrm{Sim}(G)$ and $\mathrm{Sim}(G) \to \mathrm{Coh}(G)$ constructed above are mutually inverse bijections. Let us verify that the composition $\mathrm{Coh}(G) \to \mathrm{Sim}(G) \to \mathrm{Coh}(G)$ is the identity. Consider the pair $(\widetilde{L}, i_\theta \in \mathrm{Bas}(\widetilde{L}) = H\backslash G)$ derived as above from a subgroup of finite index $H \subset G$ and a cohomology class $\theta \in H^2(H;K^*)$. The stabilizer of $i_\theta = H\backslash H$ with respect to the natural action of G on $H\backslash G$ is H. It follows from the definition of $\widetilde{L}$ that $i_\theta \widetilde{L}_\alpha = L_\alpha^\theta$ for $\alpha \in H$ and $i_\theta \widetilde{L}_\alpha = 0$ for $\alpha \in G - H$. For each $\alpha \in H$, set

$$s_\alpha = l_\alpha \in L_\alpha^\theta = i_\theta \widetilde{L}_\alpha,$$

where l_α is the vector used in the definition of L^θ. Now, it is obvious that the construction of Section 5.4 applied to $(\widetilde{L}, i_\theta)$ gives $\nabla_{i_\theta} = \theta$.

To accomplish the proof, it suffices to show that the mapping $\mathrm{Sim}(G) \to \mathrm{Coh}(G)$ is injective. It is enough to prove that a nonsingular simple crossed G-algebra (L,φ) with distinguished basic idempotent $i_0 \in I = \mathrm{Bas}(L)$ can be reconstructed (up to isomorphism) from the stabilizer $H \subset G$ of i_0 and from the H-algebra

$$i_0 L = \bigoplus_{\alpha \in H} K s_\alpha,$$

where s_α is a non-zero vector in $i_0 L_\alpha \cong K$. Since the action of G on I is transitive, for each $i \in I$ there is $\omega_i \in G$ such that $\varphi_{\omega_i}(i_0) = i$. We take $\omega_{i_0} = 1$. The isomorphism $\varphi_{\omega_i} : L \to L$ maps $i_0 L$ bijectively onto iL. Therefore the set

$$\{\varphi_{\omega_i}(s_\alpha) \mid i \in I, \alpha \in H\}$$

is a basis of $L = \bigoplus_{i \in I} iL$. The product of two basis elements is given by

$$\varphi_{\omega_i}(s_\alpha)\, \varphi_{\omega_{i'}}(s_{\alpha'}) = \begin{cases} \varphi_{\omega_i}(s_\alpha s_{\alpha'}) & \text{if } i = i', \\ 0 & \text{if } i \neq i'. \end{cases}$$

This computes L as a G-algebra. It remains to recover the action φ of G on L. For $\alpha \in H$, the homomorphism $\varphi_\alpha : i_0 L \to i_0 L$ is uniquely determined from the condition $\varphi_\alpha(s_\beta)s_\alpha = s_\alpha s_\beta \neq 0$ for $\beta \in H$, cf. (5.4.b). Each $\beta \in G$ expands uniquely as a product $\omega_i \alpha$ for some $i \in I$ and $\alpha \in H$. This allows us to recover the restriction of $\varphi_\beta = \varphi_{\omega_i}\varphi_\alpha$ to $i_0 L$. Knowing these restrictions for all $\beta \in G$, we can uniquely recover the whole action of G on L because the basis vectors of L have the form $\varphi_\beta(s)$ with $s \in i_0 L$. □

5.7 Proof of Theorem 5.5. Lemma 5.6.1 implies that $\mathrm{Sim}(G)/G = \mathrm{Coh}(G)/G$. It is clear that $\mathrm{Sim}(G)/G$ is the set of isomorphism classes of nonsingular simple crossed G-algebras. On the other hand, the formula $(H \subset G) \mapsto H\backslash G$ determines a bijection of $\mathrm{Coh}(G)/G$ onto the set of isomorphism classes of pairs (a finite transitive G-set I, an element of $H^2_G(I; K^*)$). Combining with (5.4.c) and the remarks at the beginning of Section 5.6, we obtain Theorem 5.5.

5.8 Corollary. *Any nonsingular semisimple crossed G-algebra over a field K of characteristic zero splits as a direct sum $\bigoplus_j L^{G,H_j,\theta_j}$, where j runs over a finite set of indices, H_j is a subgroup of G of finite index, and $\theta_j \in H^2(H_j; K^*)$. This splitting is unique up to replacing (H_j, θ_j) by $(\gamma_j H_j \gamma_j^{-1}, (\gamma_j)_*(\theta_j))$ for some $\gamma_j \in G$.*

5.9 Remark. The proof of Lemma 5.6.1 yields a computation of the classifying invariant of the simple crossed G-algebra $L = L^{G,H,\theta}$ associated with a subgroup H of G of finite index and $\theta \in H^2(H; K^*)$. Namely, $\mathrm{Bas}(L) = H\backslash G$ and $\nabla(L) \in H^2_G(H\backslash G, K^*)$ assigns to each right coset $i = H\gamma \in H\backslash G$ with $\gamma \in G$ the cohomology class $(\gamma^{-1})_*(\theta) \in H^2(G_i; K^*)$. Note that $\beta \in G$ acts on $H\backslash G$ via multiplication by β^{-1} on the right (cf. Section 4.2), so that the stabilizer G_i of $i = H\gamma$ is equal to $\gamma^{-1}H\gamma$.

II.6 Semisimple crossed Frobenius G-algebras

6.1 The function F_L. Consider a semisimple crossed Frobenius G-algebra L over K with inner product η. We define a map F_L from $I = \mathrm{Bas}(L)$ to K by

$$F_L(i) = \eta(i,i) = \eta(i, 1_L) \in K,$$

where $i \in I$. Thus, F_L is the restriction of the K-linear function $L_1 \to K$, $\ell \mapsto \eta(\ell, 1_L)$ to the basis I of L_1. By Section 2.1, η is entirely determined by F_L.

Clearly, $\eta(i,i') = \eta(ii', 1) = \eta(0,1) = 0$ for distinct $i, i' \in I$. The non-degeneracy of η on L_1 implies that $F_L(I) \subset K^*$. The invariance of η under the action of G on L implies the invariance of F_L under the action of G on I.

6.2 Basic triples. Let L be a semisimple crossed Frobenius G-algebra over a field K of characteristic zero. The triple $(\mathrm{Bas}(L), \nabla(L), F_L)$ is called the *basic triple* of L. For example, if $L = L^{G,H,\theta,k}$, where H is a subgroup of G of finite index, $\theta \in H^2(H; K^*)$, and $k \in K^*$, then $\mathrm{Bas}(L)$ and $\nabla(L)$ are computed in Remark 5.9 and $F_L(i) = k$ for all $i \in \mathrm{Bas}(L)$.

Two triples of the form (a finite G-set I, an element of $H^2_G(I; K^*)$, a G-invariant mapping $I \to K^*$) are *isomorphic* if there is a G-equivariant bijection between the underlying G-sets such that the induced isomorphism in cohomology carries the cohomology classes into each other and the composition with this bijection transforms the mappings to K^* into each other. There is an obvious operation of *disjoint union* on the set of isomorphism types of such triples:

$$(I_1, \nabla_1, F_1) \amalg (I_2, \nabla_2, F_2) = (I_1 \amalg I_2, \nabla_1 + \nabla_2, F),$$

where $F\colon I_1 \amalg I_2 \to K^*$ is determined by $F|_{I_1} = F_1$ and $F|_{I_2} = F_2$. It is clear that the basic triple of the direct sum of two semisimple crossed Frobenius G-algebras L_1, L_2 is a disjoint union of the basic triples of L_1, L_2.

The following theorem shows that the basic triple is a complete isomorphism invariant of a semisimple crossed Frobenius G-algebra.

6.3 Theorem. *Let K be a field of characteristic zero. The formula*

$$L \mapsto (\mathrm{Bas}(L), \nabla(L), F_L) \tag{6.3.a}$$

establishes a bijective correspondence between the isomorphism classes of semisimple crossed Frobenius G-algebras L over K and the isomorphism classes of triples (a finite G-set I, an element ∇ of $H^2_G(I; K^*)$, a G-invariant function $I \to K^*$).

Proof. The injectivity of (6.3.a) follows from the injectivity in Theorem 5.5 and the fact that the inner product of a semisimple crossed Frobenius G-algebra L is entirely determined by F_L.

Every triple (I, ∇, F) as above splits as a disjoint union of triples with transitive underlying G-sets. For a triple (I, ∇, F) with transitive I, the function F is constant and $I = H\backslash G$ for a subgroup H of G of finite index. Example 4.4 and Remark 5.9 show that such (I, ∇, F) lies in the image of the map (6.3.a). The surjectivity of (6.3.a) follows by realizing disjoint unions of triples by direct sums of semisimple crossed Frobenius G-algebras. □

6.4 Corollary. *Any semisimple crossed Frobenius G-algebra over a field K of characteristic zero splits as a direct sum $\bigoplus_j L^{G,H_j,\theta_j,k_j}$, where j runs over a finite set of indices, H_j is a subgroup of G of finite index, $\theta_j \in H^2(H_j; K^*)$, and $k_j \in K^*$. This splitting is unique up to replacing (H_j, θ_j) by $(\gamma_j H_j \gamma_j^{-1}, (\gamma_j)_*(\theta_j))$ for some $\gamma_j \in G$.*

6.5 Corollary. *If G is a free group, then there is a bijective correspondence between the isomorphism classes of semisimple crossed Frobenius G-algebras over a field K of characteristic zero and the isomorphism classes of pairs (a finite G-set I, a G-invariant function $I \to K^*$).*

Indeed, $H_G^2(I; K^*) = 0$ for a free group G and any G-set I.

6.6 Normal G-algebras. We call a semisimple crossed Frobenius G-algebra (L, η) *normal* if $\eta(i, i) = 1$ for all $i \in \mathrm{Bas}(L)$. The G-algebra L^θ associated with any $\theta \in H^2(G; K^*)$ and the G-algebra $L^{G,H,\theta}$ in Example 4.4 are normal.

6.6.1 Corollary. *Every nonsingular semisimple crossed G-algebra over a field of characteristic zero has a unique inner product turning it into a normal crossed Frobenius G-algebra.*

6.7 Example. Consider the crossed Frobenius G-algebra $(L = K[G'], \eta, \varphi)$ derived from a group epimorphism $q: G' \to G$ with finite kernel Γ lying in the center of G'; see Section 3.3.1. Assume that K is a field of characteristic zero. Then L is semisimple and the basic idempotents of L are the vectors

$$e_\rho = |\Gamma|^{-1} \sum_{h \in \Gamma} \rho(h) h \in L_1$$

labeled by the group homomorphisms $\rho: \Gamma \to K^*$. The action φ of G on $\mathrm{Bas}(L) = \{e_\rho\}_\rho$ is trivial and $F_L(e_\rho) = |\Gamma|^{-1}$ for all ρ. We can rescale the inner product η of L by multiplying it by $|\Gamma|$. This gives a normal crossed Frobenius G-algebra L^+ that splits as a direct sum of the simple normal crossed Frobenius G-algebras $\{e_\rho L^+\}_\rho$. These G-algebras can be described via the cohomology class $\theta \in H^2(G; \Gamma)$ associated with q in the standard way. Each homomorphism $\rho: \Gamma \to K^*$ transforms θ into a certain $\theta_\rho \in H^2(G; K^*)$ and $e_\rho L^+ \cong L^{\theta_\rho}$.

II.7 Hermitian G-algebras

In this section we assume that the ground ring K is endowed with a ring involution $K \to K, k \mapsto \bar{k}$.

7.1 Hermitian Frobenius G-algebras. A *Hermitian involution* on a Frobenius G-algebra $(L = \bigoplus_{\alpha\in G} L_\alpha, \eta)$ over K is an involutive antilinear antiautomorphism $L \to L$, $a \mapsto \bar{a}$ carrying each L_α into $L_{\alpha^{-1}}$ and transforming η into $\overline{\eta}$. Thus, for any $a, b \in L$, $k \in K$,

$$\bar{\bar{a}} = a, \quad \overline{ka} = \bar{k}\,\bar{a}, \quad \overline{ab} = \bar{b}\,\bar{a}, \quad \eta(\bar{a}, \bar{b}) = \overline{\eta(a, b)}. \tag{7.1.a}$$

It follows from these conditions that $\overline{1_L} = 1_L$. The term "Hermitian" is suggested by the fact that the formula $(a, b) \mapsto \eta(a, \bar{b})$ with $a, b \in L_\alpha$, defines a Hermitian form on every L_α. A Frobenius G-algebra endowed with a Hermitian involution is said to be *Hermitian*.

Observe that for any $a \in L$,

$$\overline{\eta(a, \bar{a})} = \eta(\bar{a}, \bar{\bar{a}}) = \eta(\bar{a}, a) = \eta(a, \bar{a}).$$

In particular, if $K = \mathbb{C}$ with complex conjugation then $\eta(a, \bar{a}) \in \mathbb{R}$. If $\eta(a, \bar{a}) > 0$ for all non-zero $a \in L$, then L is *unitary*.

If L is a crossed Frobenius G-algebra, then a Hermitian involution on L is required to commute with the action of G so that $\varphi_\beta(\bar{a}) = \overline{\varphi_\beta(a)}$ for any $a \in L$ and $\beta \in G$.

It is easy to check that direct sums, tensor products, pull-backs, duals, and transfers of Hermitian (crossed) Frobenius group-algebras are again Hermitian (crossed) Frobenius group-algebras. The same is true for unitary group-algebras. We can define a category, $\mathcal{HQ}(G)$ (resp. $\mathcal{UQ}(G)$), whose objects are Hermitian (resp. unitary) crossed Frobenius G-algebras. The morphisms are defined as in Section 3.2 with the additional condition that they commute with the Hermitian involutions.

7.2 Example. Let $\{\theta_{\alpha,\beta}\}_{\alpha,\beta\in G}$ be a normalized 2-cocycle of G with values in the multiplicative group $S = \{k \in K \mid k\bar{k} = 1_K\}$. By Section 3.3.2 this cocycle gives rise to a crossed Frobenius G-algebra $L = \bigoplus_{\alpha\in G} K l_\alpha$. We introduce a Hermitian involution on L. For $\alpha \in G$, set

$$\overline{l_\alpha} = \overline{\theta_{\alpha,\alpha^{-1}}}\, l_{\alpha^{-1}} = (\theta_{\alpha,\alpha^{-1}})^{-1} l_{\alpha^{-1}}.$$

This extends uniquely to an antilinear homomorphism $a \mapsto \bar{a} \colon L \to L$. We claim that it satisfies (7.1.a). The involutivity follows from the equalities

$$\theta_{\alpha,\alpha^{-1}}\overline{\theta_{\alpha^{-1},\alpha}} = \theta_{\alpha,\alpha^{-1}}\overline{\theta_{\alpha,\alpha^{-1}}} = 1_K$$

where we use the identity $\theta_{\alpha^{-1},\alpha} = \theta_{\alpha,\alpha^{-1}}$ (cf. Section 2.3.2). Similarly,

$$\begin{aligned}\eta(\overline{l_\alpha}, \overline{l_{\alpha^{-1}}}) &= \eta(\overline{\theta_{\alpha,\alpha^{-1}}}\, l_{\alpha^{-1}}, \overline{\theta_{\alpha^{-1},\alpha}}\, l_\alpha) \\ &= \overline{\theta_{\alpha,\alpha^{-1}}}\; \overline{\theta_{\alpha^{-1},\alpha}}\; \eta(l_{\alpha^{-1}}, l_\alpha) \\ &= \overline{\theta_{\alpha,\alpha^{-1}}}\; \overline{\theta_{\alpha^{-1},\alpha}}\; \theta_{\alpha^{-1},\alpha} \\ &= \overline{\theta_{\alpha,\alpha^{-1}}} \\ &= \overline{\eta(l_\alpha, l_{\alpha^{-1}})}.\end{aligned}$$

To prove that the involution $a \mapsto \bar{a}$ is an antiautomorphism we must check that $\overline{l_\alpha l_\beta} = \overline{l_\beta}\, \overline{l_\alpha}$ for any $\alpha, \beta \in G$. This is equivalent to the 5-term identity

$$(\theta_{\alpha,\beta}\, \theta_{\alpha\beta,(\alpha\beta)^{-1}})^{-1} = (\theta_{\alpha,\alpha^{-1}} \theta_{\beta,\beta^{-1}})^{-1} \theta_{\beta^{-1},\alpha^{-1}}. \tag{7.2.a}$$

To prove this identity, set $\gamma = \beta^{-1}$ in (1.2.a). This gives $\theta_{\alpha\beta,\beta^{-1}} = \theta_{\beta,\beta^{-1}} \theta_{\alpha,\beta}^{-1}$ (we use that $\theta_{\alpha,1} = 1$.) Now, replace α, β, γ in (1.2.a) with $\alpha\beta, \beta^{-1}, \alpha^{-1}$, respectively. Substituting $\theta_{\alpha\beta,\beta^{-1}} = \theta_{\beta,\beta^{-1}} \theta_{\alpha,\beta}^{-1}$ in the resulting formula, we obtain (7.2.a).

It remains to check that $\varphi_\beta(\overline{l_\alpha}) = \overline{\varphi_\beta(l_\alpha)}$. Note that $l_\alpha \overline{l_\alpha} = 1$. Therefore $\varphi_\beta(l_\alpha)\, \varphi_\beta(\overline{l_\alpha}) = 1$. This characterizes $\varphi_\beta(\overline{l_\alpha})$ as the unique element of $L_{\beta\alpha^{-1}\beta^{-1}}$ inverse to $\varphi_\beta(l_\alpha)$. We claim that $\overline{\varphi_\beta(l_\alpha)} \in L_{\beta\alpha^{-1}\beta^{-1}}$ is also inverse to $\varphi_\beta(l_\alpha)$. By (3.3.a), $\varphi_\beta(l_\alpha) = k l_{\beta\alpha\beta^{-1}}$, where $k = \theta_{\beta,\alpha} \theta^{-1}_{\beta\alpha\beta^{-1},\beta} \in S$. Then

$$\varphi_\beta(l_\alpha)\, \overline{\varphi_\beta(l_\alpha)} = k\, \bar{k}\, l_{\beta\alpha\beta^{-1}}\, \overline{l_{\beta\alpha\beta^{-1}}} = l_{\beta\alpha\beta^{-1}}\, \overline{l_{\beta\alpha\beta^{-1}}} = 1.$$

Hence $\overline{\varphi_\beta(l_\alpha)} = \varphi_\beta(\overline{l_\alpha})$.

Note that $\eta(l_\alpha, \overline{l_\alpha}) = 1$ for all α. Therefore if $K = \mathbb{C}$ with complex conjugation, then L is unitary.

Chapter III

Two-dimensional HQFTs

III.1 The underlying G-algebra

Throughout this chapter the symbol X denotes an aspherical connected CW-space with base point x and fundamental group $G = \pi_1(X, x)$. The main aim of the chapter is to classify two-dimensional X-HQFTs in terms of G-algebras.

In this section we derive from a two-dimensional X-HQFT (A, τ) the "underlying" crossed Frobenius G-algebra.

1.1 Preliminaries. By Section I.3, we can apply (A, τ) to 1-dimensional homotopy X-manifolds and 2-dimensional homotopy X-cobordisms. One-dimensional homotopy X-manifolds will be called *X-curves*. A connected non-empty X-curve M is a pointed oriented circle endowed with a homotopy class of maps to X carrying the base point to x. This homotopy class is an element $\alpha = \alpha(M)$ of $G = \pi_1(X, x)$. We shall sometimes write $M \approx S^1$ for such X-curve M and denote it (M, α). The K-module A_M depends only on α up to canonical isomorphism. This follows from Lemma I.3.3.1 and the obvious fact that any two oriented pointed circles are related by an orientation preserving and base point preserving homeomorphism unique up to isotopy.

For each $\alpha \in G$, set $L_\alpha = A_M$ for any X-curve $M \approx S^1$ representing α. Clearly, L_α is a projective K-module of finite type. For a non-connected X-curve M with components $\{(M^p, \alpha_p \in G)\}_p$, we have $A_M = \bigotimes_p L_{\alpha_p}$.

Two-dimensional homotopy X-cobordisms will be called *X-surfaces*. When the underlying surface is a disk, an annulus, or a disk with holes we call the X-surface an *X-disk*, an *X-annulus*, or an *X-disk with holes*, respectively. To transform a compact oriented surface W into an X-surface, we must orient the components of ∂W, provide them with base points, and fix a homotopy class of maps $W \to X$. (The maps are always supposed to be pointed, i.e., to carry the base points of ∂W to x. By homotopy, we mean homotopy in the class of pointed maps.) To specify the orientation on ∂W we label the components of ∂W with signs $\pm$. A component M of ∂W labeled with $+$ is denoted M_+ and is provided with orientation induced from that in W. A component M of ∂W labeled with $-$ is denoted M_- and is provided with orientation opposite to the one induced from W. In this way W becomes an X-cobordism between the X-curves $\coprod_p (M_-^p, \alpha_p)$ (the bottom base) and $\coprod_q (N_+^q, \beta_q)$ (the top base), where $\{M^p\}_p$ (resp. $\{N^q\}_q$) are the components of ∂W labeled with $-$ (resp. $+$) and $\alpha_p \in G$ (resp. $\beta_q \in G$) is represented by the restriction of the given homotopy class of maps $W \to X$ to the oriented pointed circle M_-^p (resp. N_+^q). The HQFT (A, τ) yields a vector

$$\tau(W) = \tau\big(W, \textstyle\coprod_p M_p, \coprod_q N_q, g\big)$$
$$\in \mathrm{Hom}_K\big(\textstyle\bigotimes_p L_{\alpha_p}, \bigotimes_q L_{\beta_q}\big) = \bigotimes_p L^*_{\alpha_p} \otimes \bigotimes_q L_{\beta_q}.$$

The gluing axiom (1.2.6), p. 3, can be reformulated in the language of these vectors as a *gluing rule* which we state for simplicity only for gluings along single circles: If W_0, W_1, W are X-surfaces such that W is obtained from $W_0 \amalg W_1$ by gluing a boundary circle (M_-, α) of W_0 to a boundary circle (N_+, α) of W_1 along a (homotopy) X-homeomorphism $M \approx N$, then the vector $\tau(W)$ is obtained from the tensor product $\tau(W_0) \otimes \tau(W_1)$ by the tensor contraction induced by the evaluation pairing $L^*_\alpha \otimes L_\alpha \to K$, $a \otimes b \mapsto a(b)$, where $a \in L^*_\alpha$ and $b \in L_\alpha$.

1.2 Annuli and disks with holes. We need to discuss in detail the structures of X-cobordisms on annuli and disks with two holes. Notation introduced here will be systematically used throughout this chapter.

Set $C = S^1 \times [0,1]$. We fix once and for all an orientation in C. Set $C^0 = S^1 \times 0 \subset \partial C$ and $C^1 = S^1 \times 1 \subset \partial C$. We provide C^0, C^1 with base points $(s, 0)$, $(s, 1)$, respectively, where $s \in S^1$. It is convenient to think of C as of an annulus in $\mathbb{R}^2$ with clockwise orientation such that C^0 (resp. C^1) is the internal (resp. external) boundary component and s is the bottom point of S^1; cf. Figure III.1.

Given signs $\varepsilon, \mu = \pm$, denote by $C_{\varepsilon,\mu}$ the oriented annulus C with oriented pointed boundary $C^0_\varepsilon \cup C^1_\mu$. In the category of oriented manifolds,

$$\partial C_{\varepsilon,\mu} = \varepsilon C^0_\varepsilon \cup \mu C^1_\mu.$$

For example, if $C \subset \mathbb{R}^2$ as above, then both components of ∂C_{-+} are oriented clockwise.

For $\varepsilon, \mu = \pm$, a homotopy class of (pointed) maps $g \colon C_{\varepsilon,\mu} \to X$ is determined by the elements α, β of G represented by the loops $g|_{C^0_\varepsilon}$ and $g|_{s\times[0,1]}$, respectively. Here the interval $[0,1]$ is oriented from 0 to 1. Note that the loop $g|_{C^1_\mu}$ represents $\beta^{-1}\alpha^{-\varepsilon\mu}\beta \in G$. Denote by $C_{\varepsilon,\mu}(\alpha;\beta)$ the annulus $C_{\varepsilon,\mu}$ endowed with the homotopy class of maps to X corresponding to $\alpha, \beta \in G$; cf. Figure III.1.

Let D be a 2-disk with two holes. Let Y, Z, T be the components of ∂D with base points y, z, t, respectively. We fix two proper embedded arcs ty and tz in D oriented from t to y, z and meeting solely at t. We provide D with the orientation obtained by rotating ty towards tz in D around t. It is convenient to think of D as of a disk with two holes in $\mathbb{R}^2$ with clockwise orientation such that Y, Z are the internal boundary components and T is the external boundary component.

Given $\varepsilon, \mu, \nu = \pm$, denote by $D_{\varepsilon,\mu,\nu}$ the disk with two holes D with oriented pointed boundary $Y_\varepsilon \cup Z_\mu \cup T_\nu$. In the category of oriented manifolds,

$$\partial D_{\varepsilon,\mu,\nu} = \varepsilon Y_\varepsilon \cup \mu Z_\mu \cup \nu T_\nu.$$

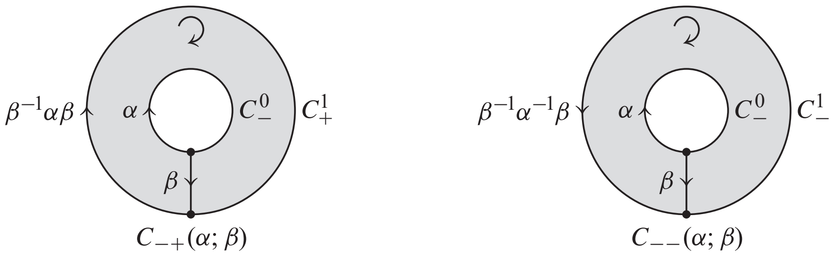


Figure III.1. The X-annuli $C_{-+}(\alpha;\beta)$ and $C_{--}(\alpha;\beta)$.

For example, if $D \subset \mathbb{R}^2$ as above, then all components of ∂D_{--+} are oriented clockwise; cf. Figure III.2.

Pick again arbitrary signs $\varepsilon, \mu, \nu = \pm$. To a homotopy class of (pointed) maps $g\colon D_{\varepsilon,\mu,\nu} \to X$ we assign the homotopy classes of the loops $g|_{Y_\varepsilon}, g|_{Z_\mu}, g|_{ty}, g|_{tz}$. This establishes a bijection $[D_{\varepsilon,\mu,\nu}, X] = G^4$. For $\alpha, \beta, \rho, \delta \in G$ we denote by $D_{\varepsilon,\mu,\nu}(\alpha,\beta;\rho,\delta)$ the disk with holes $D_{\varepsilon,\mu,\nu}$ endowed with the homotopy class of maps to X corresponding to the tuple $(\alpha,\beta,\rho,\delta) \in G^4$; cf. Figure III.2. Note that the loops $g|_{Y_\varepsilon}, g|_{Z_\mu}$, and $g|_{T_\nu}$ represent α, β, and $\gamma = (\rho\alpha^{-\varepsilon}\rho^{-1}\delta\beta^{-\mu}\delta^{-1})^\nu$, respectively. As an exercise, the reader may construct X-homeomorphisms

$$D_{\varepsilon,\mu,\nu}(\alpha,\beta;\rho,\delta) \approx D_{\mu,\nu,\varepsilon}(\beta,\gamma;\rho^{-1}\delta,\rho^{-1}) \approx D_{\nu,\varepsilon,\mu}(\gamma,\alpha;\delta^{-1},\delta^{-1}\rho).$$

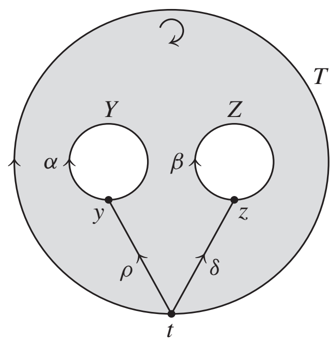


Figure III.2. The X-disk with holes $D_{--+}(\alpha,\beta;\rho,\delta)$.

1.3 Multiplication in L. We provide the K-module

$$L = \bigoplus_{\alpha\in G} L_\alpha$$

with a structure of an associative algebra over K as follows. The X-disk with two holes $D_{--+}(\alpha,\beta;1,1)$ is an X-cobordism between the X-curves $(Y_-,\alpha) \amalg (Z_-,\beta)$

and $(T_+, \alpha\beta)$. The corresponding homomorphism

$$\tau(D_{--+}(\alpha, \beta; 1, 1)) \colon L_\alpha \otimes L_\beta \to L_{\alpha\beta}$$

defines a K-bilinear multiplication in L by

$$ab = \tau(D_{--+}(\alpha, \beta; 1, 1))(a \otimes b) \in L_{\alpha\beta} \tag{1.3.a}$$

for $a \in L_\alpha$ and $b \in L_\beta$. This multiplication is associative; see Lemma 1.6 below.

The unit of L is constructed as follows. Let B_+ be an oriented 2-disk whose boundary is pointed and endowed with orientation induced from B_+. There is only one homotopy class of maps $B_+ \to X$. The corresponding homomorphism $\tau(B_+) = \tau(B_+, \emptyset, \partial B_+) \colon K \to L_1$ carries the unit $1_K \in K$ into an element of L_1 denoted 1_L. This element is a two-sided unit of L; see Lemma 1.6.

1.4 The inner product on L. For $\alpha \in G$, the X-annulus $C_{--}(\alpha; 1)$ is an X-cobordism between $(C_-^0, \alpha) \amalg (C_-^1, \alpha^{-1})$ and $\emptyset$. Set

$$\eta_\alpha = \tau(C_{--}(\alpha; 1)) \colon L_\alpha \otimes L_{\alpha^{-1}} \to K.$$

Note that the X-annulus $C_{--}(\alpha^{-1}; 1)$ is X-homeomorphic to $C_{--}(\alpha; 1)$ via a homeomorphism permuting the boundary components. Axiom (1.2.4), p. 3, implies that $\eta_{\alpha^{-1}}$ is obtained from η_α by the permutation of the tensor factors.

We define $\eta \colon L \otimes L \to K$ to be the bilinear form whose restriction to $L_\alpha \otimes L_\beta$ is zero if $\alpha\beta \neq 1$ and is η_α if $\alpha\beta = 1$, where $\alpha, \beta \in G$. It is clear that the form η is symmetric.

1.5 The action of G on L. For $\alpha, \beta \in G$, the X-annulus $C_{-+}(\alpha; \beta^{-1})$ is an X-cobordism between (C_-^0, α) and $(C_+^1, \beta\alpha\beta^{-1})$. Set

$$\varphi_\beta = \tau(C_{-+}(\alpha; \beta^{-1})) \colon L_\alpha \to L_{\beta\alpha\beta^{-1}}.$$

By Axiom (1.2.7), p. 3, $\varphi_1 = \mathrm{id}$. For $\alpha, \beta, \gamma \in G$, the gluing of $C_{-+}(\alpha; \beta^{-1})$ to $C_{-+}(\beta\alpha\beta^{-1}; \gamma^{-1})$ yields $C_{-+}(\alpha; \beta^{-1}\gamma^{-1})$. By Axiom (1.2.6), p. 3, $\varphi_{\gamma\beta} = \varphi_\gamma\varphi_\beta$.

1.6 Lemma. *The algebra L with bilinear form η and action φ of G is a crossed Frobenius G-algebra.*

Proof. Let us prove that $(ab)c = a(bc)$ for any $a \in L_\alpha, b \in L_\beta, c \in L_\gamma$, where $\alpha, \beta, \gamma \in G$. Consider the X-surfaces

$$W_0 = D_{--+}(\alpha, \beta; 1, 1) \amalg C_{-+}(\gamma; 1) \quad \text{and} \quad W_1 = D_{--+}(\alpha\beta, \gamma; 1, 1).$$

Recall that the boundary components of an X-surface labeled with $+$ form the top base while the boundary components labeled with $-$ form the bottom base. In particular, W_0 is an X-cobordism between

$$(Y_-, \alpha) \amalg (Z_-, \beta) \amalg (C_-^0, \gamma) \quad \text{and} \quad (T_+, \alpha\beta) \amalg (C_+^1, \gamma).$$

By Axioms (1.2.5) and (1.2.7), p. 3, and the definition of multiplication in L, the homomorphism

$$\tau(W_0)\colon L_\alpha \otimes L_\beta \otimes L_\gamma \to L_{\alpha\beta} \otimes L_\gamma$$

carries $a \otimes b \otimes c$ into $ab \otimes c$. The homomorphism $\tau(W_1)\colon L_{\alpha\beta} \otimes L_\gamma \to L_{\alpha\beta\gamma}$ carries $ab \otimes c$ into $(ab)c$. The gluing of W_0 to W_1 along the obvious X-homeomorphism of the top base $(T_+, \alpha\beta) \amalg (C^1_+, \gamma)$ of W_0 on the bottom base $(Y_-, \alpha\beta) \amalg (Z_-, \gamma)$ of W_1 yields an X-surface W, see Figure III.3. By Axiom (1.2.6), p. 3,

$$\tau(W)(a \otimes b \otimes c) = \tau(W_1)\,\tau(W_0)(a \otimes b \otimes c) = \tau(W_1)(ab \otimes c) = (ab)c.$$

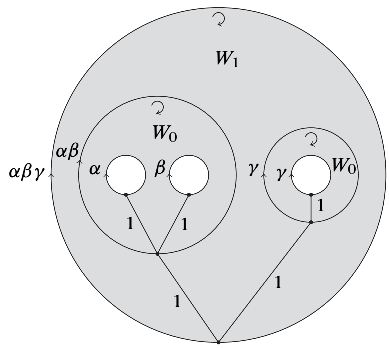


Figure III.3. The X-disk with three holes W.

The same X-surface W can be obtained by gluing the X-surfaces

$$W_2 = C_{-+}(\alpha; 1) \amalg D_{--+}(\beta, \gamma; 1, 1) \quad \text{and} \quad W_3 = D_{--+}(\alpha, \beta\gamma; 1, 1)$$

along an X-homeomorphism

$$(C^1_+, \alpha) \amalg (T_+, \beta\gamma) \approx (Y_-, \alpha) \amalg (Z_-, \beta\gamma),$$

see Figure III.4. Therefore $\tau(W)(a \otimes b \otimes c) = a(bc)$. This gives $(ab)c = a(bc)$.

Let us prove that 1_L is a right unit of L. For $\alpha \in G$, consider the X-surfaces

$$W_0 = C_{-+}(\alpha; 1) \amalg B_+ \quad \text{and} \quad W_1 = D_{--+}(\alpha, 1; 1, 1).$$

The gluing of W_0 to W_1 along an X-homeomorphism

$$(C^1_+, \alpha) \amalg (\partial B_+, 1) \approx (Y_-, \alpha) \amalg (Z_-, 1)$$

yields an X-surface X-homeomorphic to $C_{-+}(\alpha; 1)$. By Axioms (1.2.7) and (1.2.6), p. 3, for any $a \in L_\alpha$,

$$a = \tau(C_{-+}(\alpha; 1))(a) = \tau(W_1)\,\tau(W_0)(a) = \tau(W_1)(a \otimes 1_L) = a1_L.$$

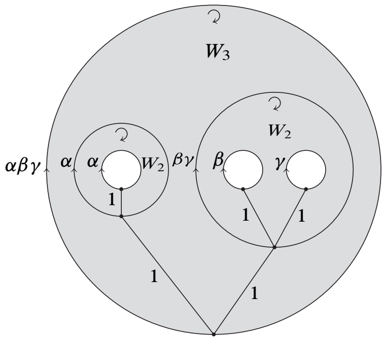


Figure III.4. Another splitting of W.

The proof that 1_L is a left unit of L is similar.

We claim that the triple (L, η, φ) satisfies Axioms (2.1.1)–(2.1.3) and (3.1.1)–(3.1.4) (p. 24, 25), of a crossed Frobenius G-algebra. Axiom (2.1.1) is obvious, and (2.1.2) directly follows from the definitions and Lemma I.1.3.1. We verify the remaining axioms.

(2.1.3): Let us prove that $\eta(ab, c) = \eta(a, bc)$ for any $a \in L_\alpha, b \in L_\beta, c \in L_\gamma$, where $\alpha, \beta, \gamma \in G$. If $\alpha\beta\gamma \neq 1$, then $\eta(ab, c) = 0 = \eta(a, bc)$. Assume that $\alpha\beta\gamma = 1$. Gluing the X-annulus $C_{--}(\alpha\beta; 1)$ to $D_{--+}(\alpha, \beta; 1, 1)$ along an X-homeomorphism $(C_-^0, \alpha\beta) \approx (T_+, \alpha\beta)$, we obtain $D_{---}(\alpha, \beta; 1, 1)$. Therefore

$$\eta(ab, c) = \tau(D_{---}(\alpha, \beta; 1, 1))(a \otimes b \otimes c),$$

where

$$\tau(D_{---}(\alpha, \beta; 1, 1)) \in \mathrm{Hom}_K(L_\alpha \otimes L_\beta \otimes L_\gamma, K).$$

Similarly, gluing $C_{--}(\alpha^{-1}; 1)$ to $D_{--+}(\beta, \gamma; 1, 1)$ along an X-homeomorphism $(C_-^0, \alpha^{-1}) \approx (T_+, \beta\gamma)$, we obtain $D_{---}(\beta, \gamma; 1, 1)$. Hence

$$\eta(a, bc) = \tau(D_{---}(\beta, \gamma; 1, 1))(b \otimes c \otimes a),$$

where

$$\tau(D_{---}(\beta, \gamma; 1, 1)) \in \mathrm{Hom}_K(L_\beta \otimes L_\gamma \otimes L_\alpha, K).$$

It remains to observe that there is an X-homeomorphism

$$D_{---}(\alpha, \beta; 1, 1) \approx D_{---}(\beta, \gamma; 1, 1)$$

mapping the boundary components Y, Z, T of the first X-disk with holes onto the boundary components T, Y, Z of the second X-disk with holes, respectively. By Axiom (1.2.4), $\eta(ab, c) = \eta(a, bc)$.

(3.1.1): We need to prove only that $\varphi_\gamma(ab) = \varphi_\gamma(a)\,\varphi_\gamma(b)$ for any $a \in L_\alpha$ and $b \in L_\beta$, where $\alpha, \beta, \gamma \in G$. Gluing $C_{-+}(\alpha\beta; \gamma^{-1})$ to $D_{--+}(\alpha, \beta; 1, 1)$ along an X-homeomorphism $(C_-^0, \alpha\beta) \approx (T_+, \alpha\beta)$, we obtain $D_{--+}(\alpha, \beta; \gamma, \gamma)$, see Figure III.5.

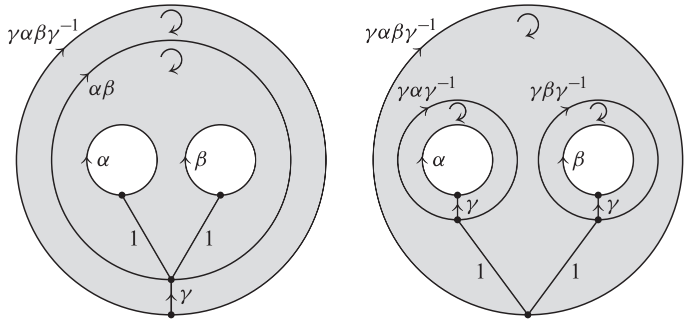


Figure III.5. Two splittings of $D_{--+}(\alpha, \beta; \gamma, \gamma)$.

Hence

$$\varphi_\gamma(ab) = \tau(D_{--+}(\alpha, \beta; \gamma, \gamma))(a \otimes b),$$

where

$$\tau(D_{--+}(\alpha, \beta; \gamma, \gamma)) \in \mathrm{Hom}_K(L_\alpha \otimes L_\beta, L_{\gamma\alpha\beta\gamma^{-1}}).$$

Gluing $C_{-+}(\alpha; \gamma^{-1}) \amalg C_{-+}(\beta; \gamma^{-1})$ to $D_{--+}(\gamma\alpha\gamma^{-1}, \gamma\beta\gamma^{-1}; 1, 1)$ along

$$(C^1_+, \gamma\alpha\gamma^{-1}) \amalg (C^1_+, \gamma\beta\gamma^{-1}) \approx (Y_-, \gamma\alpha\gamma^{-1}) \amalg (Z_-, \gamma\beta\gamma^{-1}),$$

we also obtain $D_{--+}(\alpha, \beta; \gamma, \gamma)$. Therefore

$$\varphi_\gamma(a)\, \varphi_\gamma(b) = \tau(D_{--+}(\alpha, \beta; \gamma, \gamma))(a \otimes b) = \varphi_\gamma(ab).$$

(3.1.2): Note first that for any $\alpha, \beta, \rho, \delta \in G$, the homomorphism

$$\tau(D_{--+}(\alpha, \beta; \rho, \delta)) \colon L_\alpha \otimes L_\beta \to L_{\rho\alpha\rho^{-1}\delta\beta\delta^{-1}}$$

can be computed in terms of φ and multiplication in L: for $a \in L_\alpha$ and $b \in L_\beta$,

$$\tau(D_{--+}(\alpha, \beta; \rho, \delta))(a \otimes b) = \varphi_\rho(a)\, \varphi_\delta(b).$$

This directly follows from the gluing axiom (1.2.6) using the obvious splitting of $D_{--+}(\alpha, \beta; \rho, \delta)$ into $C_{-+}(\alpha; \rho^{-1}) \amalg C_{-+}(\beta; \delta^{-1})$ and $D_{--+}(\rho\alpha\rho^{-1}, \rho\beta\rho^{-1}; 1, 1)$.

Consider a self-homeomorphism f of the disk with two holes D which is the identity on T and permutes (Y, y) and (Z, z). We choose f so that $f(tz) = ty$ and $f(ty)$ is an embedded arc leading from t to z and homotopic to the product of the arcs $ty, \partial Y, (ty)^{-1}, tz$. An easy computation shows that f is an X-homeomorphism from $D_{--+}(\alpha, \beta; 1, 1)$ to $D_{--+}(\beta, \alpha; 1, \beta^{-1})$. Axiom (1.2.4) implies that the homomorphisms

$$\tau(D_{--+}(\alpha, \beta; 1, 1)) \colon L_\alpha \otimes L_\beta \to L_{\alpha\beta}$$

and

$$\tau(D_{--+}(\beta, \alpha; 1, \beta^{-1})) \colon L_\beta \otimes L_\alpha \to L_{\alpha\beta}$$

are obtained from each other by the flip $L_\alpha \otimes L_\beta \to L_\beta \otimes L_\alpha$. Therefore for any $a \in L_\alpha, b \in L_\beta$,

$$ab = \tau(D_{--+}(\alpha,\beta;1,1))(a \otimes b) = \tau(D_{--+}(\beta,\alpha;1,\beta^{-1}))(b \otimes a) = b\,\varphi_{\beta^{-1}}(a).$$

This is equivalent to (3.1.2), p. 25.

(3.1.3): A Dehn twist about the circle $S^1 \times (1/2) \subset C_{-+}(\alpha;1)$ defines an X-homeomorphism $C_{-+}(\alpha;1) \to C_{-+}(\alpha;\alpha^{-1})$. It follows from Axiom (1.2.4) that $\tau(C_{-+}(\alpha;\alpha^{-1})) = \tau(C_{-+}(\alpha;1)) = \mathrm{id}_{L_\alpha}$. Therefore $\varphi_\alpha|_{L_\alpha} = \mathrm{id}$.

(3.1.4): Fix an orientation of S^1 and a point $s \in S^1$. Let P be the punctured torus obtained from $S^1 \times S^1$ by removing a small open 2-disk disjoint from $S^1 \times \{s\}$ and $\{s\} \times S^1$. We assume that the circle ∂P meets $S^1 \times \{s\}$ and $\{s\} \times S^1$ precisely at the point (s,s). We take (s,s) as the base point of ∂P and provide P with orientation induced from the product orientation in $S^1 \times S^1$.

Fix a map $g\colon P \to X = K(G,1)$ such that $g(s,s) = x$ and the restrictions of g to the loops $S^1 \times \{s\}$ and $\{s\} \times S^1$ represent $\alpha, \beta \in G$, respectively, see Figure III.6. (The orientations of $S^1 \times \{s\}$ and $\{s\} \times S^1$ are induced by that of S^1.)

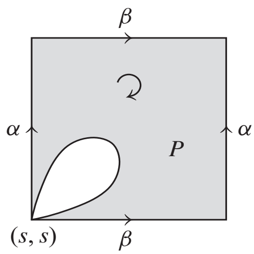


Figure III.6. The punctured X-torus P.

Then the loop $g|_{\partial P}\colon (\partial P)_- = -\partial P \to X$ represents $\alpha\beta\alpha^{-1}\beta^{-1}$, and (P,g) is an X-cobordism between $((\partial P)_-, g|_{\partial P})$ and $\emptyset$. Consider now the corresponding homomorphism $\tau(P,g)\colon L_{\alpha\beta\alpha^{-1}\beta^{-1}} \to K$. We claim that both sides of formula (3.1.a) in Chapter II are equal to $\tau(P,g)(c)$. This would imply (3.1.a).

The X-surface (P,g) can be obtained from $D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1},\alpha;1,\beta)$ by gluing the boundary circles (Z_-,α) and (T_+,α) along an X-homeomorphism. (These circles give $S^1 \times \{s\} \subset P$.) As a consequence of Lemma I.1.3.3, the homomorphism $\tau(P,g)\colon L_{\alpha\beta\alpha^{-1}\beta^{-1}} \to K$ is the partial trace of the homomorphism

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1},\alpha;1,\beta))\colon L_{\alpha\beta\alpha^{-1}\beta^{-1}} \otimes L_\alpha \to L_\alpha = K \otimes L_\alpha$$

with respect to the identity map $\mathrm{id}\colon L_\alpha \to L_\alpha$. For $d \in L_\alpha$,

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1},\alpha;1,\beta))(c \otimes d) = c\,\varphi_\beta(d) = \mu_c\,\varphi_\beta(d).$$

Therefore $\tau(P,g)(c) = \mathrm{Tr}(\mu_c\,\varphi_\beta\colon L_\alpha \to L_\alpha)$.

The same X-surface (P, g) can be obtained from the disk with two holes

$$D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1})$$

by gluing the boundary circles (Z_-, β) and (T_+, β) along an X-homeomorphism. (These circles give $\{s\} \times S^1 \subset P$.) Thus, $\tau(P, g)\colon L_{\alpha\beta\alpha^{-1}\beta^{-1}} \to K$ is the partial trace of the homomorphism

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1}))\colon L_{\alpha\beta\alpha^{-1}\beta^{-1}} \otimes L_\beta \to L_\beta = K \otimes L_\beta$$

with respect to the identity map $\mathrm{id}\colon L_\beta \to L_\beta$. For $d \in L_\beta$,

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1}))(c \otimes d) = \varphi_{\alpha^{-1}}(c)\, \varphi_{\alpha^{-1}}(d) = \varphi_{\alpha^{-1}}(cd).$$

Therefore $\tau(P, g)(c) = \mathrm{Tr}(\varphi_{\alpha^{-1}} \mu_c\colon L_\beta \to L_\beta)$. This proves (3.1.a).

Finally, the identity $\eta(\varphi_\beta(a), \varphi_\beta(b)) = \eta(a, b)$ for $a \in L_\alpha, b \in L_{\alpha^{-1}}$ follows from the fact that the X-annulus $C_{--}(\alpha; 1)$ used to define η_α can be obtained by gluing the X-annuli $C_{-+}(\alpha; \beta^{-1})$, $C_{--}(\beta\alpha\beta^{-1}; 1)$, and $C_{-+}(\alpha^{-1}; \beta^{-1})$. □

1.7 The underlying G-algebra. The crossed Frobenius G-algebra (L, η, φ) constructed above is called the *underlying G-algebra* of the X-HQFT (A, τ). We formulate here several useful properties of the underlying G-algebra.

Observe first that the 2-sphere S^2 admits only one (trivial) homotopy class of maps $g\colon S^2 \to X$. We can compute $\tau(S^2, g)$ in terms of L and η. Indeed, (S^2, g) can be obtained by gluing the X-annulus $C_{--}(1; 1)$ to a disjoint union of two copies of B_+. This implies the equality

$$\tau(S^2, g) = \eta(1_L, 1_L). \tag{1.7.a}$$

1.7.1 Lemma. *Let (A^1, τ^1) and (A^2, τ^2) be two-dimensional X-HQFTs with underlying crossed Frobenius G-algebras L^1 and L^2, respectively. Then*

(a) *the underlying crossed Frobenius G-algebra of $(A^1, \tau^1) \oplus (A^2, \tau^2)$ is $L^1 \oplus L^2$;*

(b) *the underlying crossed Frobenius G-algebra of $(A^1, \tau^1) \otimes (A^2, \tau^2)$ is $L^1 \otimes L^2$.*

Proof. This lemma follows directly from the definitions. □

1.7.2 Lemma. *If a two-dimensional X-HQFT (A', τ') is obtained from a two-dimensional X-HQFT (A, τ) by k-rescaling with $k \in K^*$, then the underlying crossed Frobenius G-algebra of (A', τ') is obtained from the underlying crossed Frobenius G-algebra of (A, τ) by k-rescaling.*

Proof. The lemma follows from the fact that for an X-surface (W, M, N) of type $D_{--+}, B_+, C_{-+}, C_{--}$, the number $\chi(W) + b_0(M) - b_0(N)$ is equal to $-1+2-1 = 0$, $1 + 0 - 1 = 0$, $0 + 1 - 1 = 0$, $0 + 2 - 1 = 1$, respectively. □

1.8 Functoriality. A morphism $\rho\colon (A,\tau) \to (A',\tau')$ of two-dimensional X-HQFTs, as defined in Section I.1.4, yields for each $\alpha \in G$ a K-isomorphism $\rho_\alpha\colon L_\alpha \to L'_\alpha$, where L and L' are the underlying G-algebras of (A,τ) and (A',τ'), respectively. The commutativity of the natural square diagrams associated with the X-cobordisms $D_{--+}(\alpha,\beta;1,1)$, B_+, $C_{--}(\alpha;1)$, and $C_{-+}(\alpha;\beta^{-1})$ implies that the direct sum $\bigoplus_\alpha \rho_\alpha\colon L \to L'$ is a morphism in the category of crossed Frobenius G-algebras $\mathcal{Q}(G)$. This defines a functor from the category of two-dimensional X-HQFTs $Q_2(X)$ to $\mathcal{Q}(G)$.

1.9 Exercises. 1. Prove that if a crossed Frobenius G-algebra L underlies a two-dimensional X-HQFT (A,τ), then the dual crossed Frobenius G-algebra $\overline{L}$ underlies the dual X-HQFT $(\overline{A},\overline{\tau})$.

2. Let X' be an Eilenberg–MacLane space of type $K(G',1)$, where G' is a group. Let (A',τ') be the two-dimensional X'-HQFT obtained from a two-dimensional X-HQFT (A,τ) by pulling back along a (pointed) map $f\colon X' \to X$; see Section I.1.4. Prove that the underlying crossed Frobenius G-algebra of (A',τ') is the pull-back of the underlying crossed Frobenius G-algebra of (A,τ) along the homomorphism $G' \to G$ induced by f, cf. Section 3.5.

III.2 Computation for cohomological HQFTs

By Sections I.2.1 and II.3.3.2, any $\theta \in H^2(G;K^*) = H^2(X;K^*)$ determines a primitive cohomological 2-dimensional X-HQFT (A^θ,τ^θ) and a crossed Frobenius G-algebra L^θ. The following theorem computes the crossed Frobenius G-algebra underlying (A^θ,τ^θ).

2.1 Theorem. *For all $\theta \in H^2(G;K^*)$, the underlying crossed Frobenius G-algebra of (A^θ,τ^θ) is isomorphic to $L^{-\theta}$.*

Proof. Set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with clockwise orientation and base point $-i$. For each $\alpha \in G = \pi_1(X,x)$, fix a loop $u_\alpha\colon S^1 \to X$ carrying $-i$ to x and representing α. We choose u_1 to be the constant loop at x. Let $p\colon [0,1] \to S^1$ be the map carrying $t \in [0,1]$ to $-i\,\exp(-2\pi i t) \in S^1$. Let $\Delta \subset \mathbb{R}^3$ be the standard 2-simplex with the vertices $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$. For $\alpha,\beta \in G$, pick a map $f_{\alpha,\beta}\colon \Delta \to X$ such that for all $t \in [0,1]$,

$$f_{\alpha,\beta}((1-t)v_0 + tv_1) = u_\alpha p(t), \quad f_{\alpha,\beta}((1-t)v_1 + tv_2) = u_\beta p(t),$$
$$f_{\alpha,\beta}((1-t)v_0 + tv_2) = u_{\alpha\beta}\, p(t).$$

We represent $\theta \in H^2(G;K^*) = H^2(X;K^*)$ by a K^*-valued singular 2-cocycle Θ on X. For any $\alpha,\beta \in G$, let $\theta_{\alpha,\beta} = \Theta(f_{\alpha,\beta}) \in K^*$ be the evaluation of Θ on the singular simplex $f_{\alpha,\beta}$. Observe that $\theta_{\alpha,\beta}$ does not depend on the choice of $f_{\alpha,\beta}$. Indeed, if $f'_{\alpha,\beta}: \Delta \to X$ is another map as above, then the formal difference $f_{\alpha,\beta} - f'_{\alpha,\beta}$ is a singular 2-cycle in X. The homology class of this 2-cycle is trivial because it can be realized by a map $S^2 \to X$ and $\pi_2(X) = 0$. Therefore $\Theta(f_{\alpha,\beta}) = \Theta(f'_{\alpha,\beta})$. A similar argument shows that $\{\theta_{\alpha,\beta}\}_{\alpha,\beta\in G}$ is a 2-cocycle. Multiplying, if necessary, Θ by a coboundary, we can ensure that $\theta_{1,1} = 1$. Both Θ and $\{\theta_{\alpha,\beta}\}_{\alpha,\beta\in G}$ represent $\theta \in H^2(G; K^*)$ and $\theta_- = \{\theta^{-1}_{\alpha,\beta}\}_{\alpha,\beta\in G}$ represents $-\theta$.

Let L be the G-algebra underlying the HQFT (A^Θ, τ^Θ). We prove that L is isomorphic to the G-algebra L^{θ_-} determined by the cocycle θ_-.

For $\alpha \in G$, the pair (S^1, u_α) is an X-curve denoted M_α. The singular 1-simplex $p: [0,1] \to S^1$ is a fundamental cycle of M_α. By definition of (A^Θ, τ^Θ) and L, this cycle determines a generating vector $p_\alpha = \langle p \rangle$ in $L_\alpha = A^\Theta_{M_\alpha} \cong K$. We claim that $p_\alpha p_\beta = \theta^{-1}_{\alpha,\beta}\, p_{\alpha\beta}$ for all $\alpha, \beta \in G$. To compute $p_\alpha p_\beta$, we apply τ^Θ to the disk with holes $D = D_{--+}(\alpha, \beta; 1, 1)$ viewed as an X-cobordism between $M_\alpha \amalg M_\beta$ and $M_{\alpha\beta}$. This gives $p_\alpha p_\beta = k\, p_{\alpha\beta}$ for $k = g^*(\Theta)(B)$, where $g: D \to X$ is the map determined by the tuple $(\alpha, \beta; 1, 1)$ and $B \in C_2(D; \mathbb{Z})$ is a fundamental singular 2-chain in D such that $\partial B = p_{\alpha\beta} - p_\alpha - p_\beta$. Recall the segments $ty, tz \subset D$. Clearly, $D' = D/ty \cup tz$ is an oriented triangle with oriented edges and identified vertices. Deforming g in its homotopy class, we may assume that $g(ty \cup tz) = x$. Then g expands as the composition of the projection $q: D \to D'$ with a map $g': D' \to X$, and k is the evaluation of $(g')^*(\Theta)$ on $q_*(B)$. Instead of $q_*(B)$ we can use more general 2-chains in D'. Namely, let $\partial D' = q(\partial D)$ be the union of the sides of D'. Then $k = (g')^*(\Theta)(B')$ for any 2-chain B' in D' such that $\partial B' = qp_{\alpha\beta} - qp_\alpha - qp_\beta$ and the image of B in $C_2(D', \partial D'; \mathbb{Z})$ represents the generator of $H_2(D', \partial D'; \mathbb{Z}) = \mathbb{Z}$ determined by the orientation of D'. There is an obvious projection $f: \Delta \to D'$ identifying the vertices of Δ and such that $g' f = f_{\alpha,\beta}$ (up to homotopy rel $\partial\Delta$). The singular chain $B' = -f$ satisfies all the conditions above. Therefore

$$k = (g')^*(\Theta)(B') = ((g')^*(\Theta)(f))^{-1} = (\Theta(g'f))^{-1} = (\Theta(f_{\alpha,\beta}))^{-1} = \theta^{-1}_{\alpha,\beta}.$$

By definition of L^{θ_-}, the formula $p_\alpha \mapsto \ell_\alpha$ for $\alpha \in G$ defines an isomorphism of G-algebras $L \cong L^{\theta_-}$. It remains to compare the inner products and the actions of G. The inner product, η^-, on L^{θ_-} satisfies $\eta^-(1_L, 1_L) = 1$. By (1.7.a), the inner product η on L satisfies $\eta(1_L, 1_L) = \tau^\Theta(S^2) = \theta(0) = 1$. Since $L_1 = L_1^{\theta_-} = K$, we have $\eta^- = \eta$. The isomorphism $L \cong L^{\theta_-}$ commutes with the action of G because there is only one action of G on L^{θ_-} satisfying the axioms of a crossed G-algebra, cf. Section II.3.3.2. □

2.2 Corollary. *Let $H \subset G$ be a subgroup of finite index and $\theta \in H^2(H; K^*)$.*

(a) *The underlying crossed Frobenius G-algebra of the cohomological two-dimensional X-HQFT $(A^{G,H,\theta}, \tau^{G,H,\theta})$ is isomorphic to $L^{G,H,-\theta}$ (see Sections* I.3.1 *and* II.4.4 *for notation).*

(b) *More generally, for $k \in K^*$, the underlying crossed Frobenius G-algebra of the rescaled cohomological two-dimensional X-HQFT $(A^{G,H,\theta}, \tau^{G,H,\theta,k})$ is isomorphic to $L^{G,H,-\theta,k}$.*

Proof. By Lemma 1.7.2, it is enough to prove (a). Let $p\colon \widetilde{X} \to X$ be the covering corresponding to $H \subset G$. Set $Y = \widetilde{X}/p^{-1}(x)$ and consider the Y-HQFT (A^θ, τ^θ) determined by

$$\theta \in H^2(H; K^*) = H^2(\widetilde{X}; K^*) = H^2(Y; K^*).$$

By definition, the X-HQFT $(A' = A^{G,H,\theta}, \tau' = \tau^{G,H,\theta})$ is the transfer of (A^θ, τ^θ). We compute the underlying crossed Frobenius G-algebra (L', η', φ') of (A', τ'). Pick a point $\tilde{x} \in p^{-1}(x)$, and for each $y \in p^{-1}(x)$, fix a path in $\widetilde{X}$ from $\tilde{x}$ to y. Let $\omega_y \in G = \pi_1(X, x)$ be the homotopy class of the loop obtained by projecting this path to X. Then $\{\omega_y\}_{y \in p^{-1}(x)}$ is a set of representatives of the right H-cosets in G. For an X-curve $M = (M, g\colon M \to X)$, the K-module A'_M is a direct sum of copies of K numerated by the lifts of g to $\widetilde{X}$. If $M \approx S^1$ and $g\colon M \to X$ represents $\alpha \in G$, then the lifts of g to $\widetilde{X}$ are numerated by $y \in p^{-1}(x)$ such that $\omega_y \alpha \omega_y^{-1} \in H$. Thus, $L'_\alpha = L'_M$ is the direct sum of copies of K numerated by such y. Let $(L, \eta, \varphi) \cong L^{-\theta}$ be the underlying crossed Frobenius H-algebra of (A^θ, τ^θ). Comparing with the transfer $(\widetilde{L}, \tilde{\eta}, \widetilde{\varphi})$ of (L, η, φ) determined by the same $\{\omega_y\}_y$, we obtain that $L' = \widetilde{L}$ as G-graded K-modules. Multiplication in L' is induced by the maps $D_{--+} \to X$ carrying the segments $ty, tz \subset D_{--+}$ to the base point x. A lift of such a map to $\widetilde{X}$ carries $ty \cup tz$ to one and the same point of $p^{-1}(x)$ and induces multiplication in the corresponding copy of K. Therefore, multiplication in L' is a direct sum of multiplications in the copies of K above. Hence $L' = \widetilde{L}$ as G-graded algebras. That $\eta' = \tilde{\eta}$ is proven similarly. The definition of the transfer implies that for any $\beta \in G$, both φ'_β and $\widetilde{\varphi}_\beta$ carry $K = L_{\omega_i \alpha \omega_i^{-1}} \subset L'_\alpha = \widetilde{L}_\alpha$ to $K = L_{\omega_{\beta(i)} \beta\alpha\beta^{-1} \omega_{\beta(i)}^{-1}} \subset L'_{\beta\alpha\beta^{-1}} = \widetilde{L}_{\beta\alpha\beta^{-1}}$ in the notation of Section II.4.2. The equality $\varphi'_\beta = \widetilde{\varphi}_\beta$ follows from the uniqueness of the action of G on L satisfying the axioms of a crossed G-algebra, cf. Section II.3.3.2. □

2.3 Corollary. *Let (A, τ) be a two-dimensional X-HQFT and L the underlying crossed Frobenius G-algebra.*

(a) *If (A, τ) is cohomological, then L is simple and normal.*
(b) *If (A, τ) is rescaled cohomological, then L is simple.*
(c) *If (A, τ) is semi-cohomological, then L is semisimple.*

Proof. Claim (a) follows from Corollary 2.2 because L^θ is simple and normal for all θ and the same is true for the transfers of L^θ. Claim (b) follows from (a). Claim (c) follows from (b) and Lemma 1.7.1 (a). □

III.3 Equivalence of categories

We state the main theorem of this chapter. Recall that $Q_2(X) = Q_2(K(G,1))$ is the category of two-dimensional X-HQFTs (see Section I.1.4) and $\mathcal{Q}(G)$ is the category of crossed Frobenius G-algebras (see Section II.3.2).

3.1 Theorem. *The functor $Q_2(X) \to \mathcal{Q}(G)$ assigning to a 2-dimensional X-HQFT its underlying crossed Frobenius G-algebra is an equivalence of categories.*

Theorem 3.1 will be proven in Section 4. Note that in Theorem 3.1 the ground ring K of the X-HQFTs and the G-algebras is an arbitrary commutative ring. For $G = \{1\}$, this theorem yields the well known equivalence between the category of 2-dimensional TQFTs and the category of commutative Frobenius algebras; see [Di], [Ab], [Kock].

3.2 Corollary. *The functor of Theorem 3.1 induces a bijective correspondence between the isomorphism classes of two-dimensional X-HQFTs and the isomorphism classes of crossed Frobenius G-algebras.*

3.3 Corollary. *The group of automorphisms of a two-dimensional X-HQFT is isomorphic to the group of automorphisms of the underlying crossed Frobenius G-algebra.*

3.4 Corollary. *Let (A, τ) be a two-dimensional X-HQFT over a field of characteristic zero. Let L be the crossed Frobenius G-algebra underlying (A, τ).*

(a) *The HQFT (A, τ) is isomorphic to a cohomological X-HQFT if and only if L is simple and normal.*

(b) *The HQFT (A, τ) is isomorphic to a rescaled cohomological X-HQFT if and only if L is simple.*

(c) *The HQFT (A, τ) is semi-cohomological if and only if L is semisimple.*

(d) *If (A, τ) is semi-cohomological, then its decomposition as a direct sum of rescaled cohomological X-HQFTs is unique.*

Proof. The necessity of the conditions on L in (a)–(c) follows from Corollary 2.3. If L is simple, then by Corollary II.6.4, L is isomorphic to $L^{G,H,\theta,k}$, where H is a subgroup of G of finite index, $\theta \in H^2(H; K^*)$, and $k \in K^*$. Corollary 2.2 implies that the crossed Frobenius G-algebras underlying the X-HQFTs (A, τ) and $(A^{G,H,\theta}, \tau^{G,H,\theta,k})$ are isomorphic. By Corollary 3.2, these X-HQFTs are isomorphic. This proves the sufficiency of the condition on L in (b). Similar arguments prove the sufficiency in (a) and (c). Claim (d) follows from Corollary 3.2 and the fact that the decomposition of a semisimple crossed Frobenius G-algebra as a direct sum of simple G-algebras is unique. □

3.5 Computation for semi-cohomological HQFTs. Consider in more detail a two-dimensional semi-cohomological X-HQFT (A, τ) over a field K of characteristic zero.

Let L be the underlying crossed Frobenius G-algebra of (A, τ). By Corollary 2.3 (c), L is semisimple, and we can consider its basic triple

$$(I = \mathrm{Bas}(L), \quad \nabla(L) \in H^2_G(I; K^*), \quad F_L \colon I \to K^*).$$

The following theorem computes the values of τ on certain closed X-surfaces.

3.5.1 Theorem. *Let W be a closed connected oriented surface, $w \in W$, and $\pi = \pi_1(W, w)$. Let $\tilde{g} \colon (W, w) \to (X, x)$ be a map inducing a group epimorphism $g \colon \pi \to G$. Then*

$$\tau(W, \tilde{g}) = \sum_{i \in I_0} (F_L(i))^{\chi(W)/2}\, g^*(-\nabla_i)([W]), \tag{3.5.a}$$

where $I_0 \subset I$ is the fixed point set of the action of G on I and $\nabla_i \in H^2(G; K^*)$ is the cohomology class defined in Section II.5.4.

Here $g^*(-\nabla_i) = -g^*(\nabla_i) \in H^2(\pi; K^*)$ and

$$g^*(-\nabla_i)([W]) = g^*(\nabla_i)(-[W]) = g^*(\nabla_i)([-W]) = \big(g^*(\nabla_i)([W])\big)^{-1} \in K^*$$

is the evaluation of $g^*(-\nabla_i)$ on the fundamental class $[W] \in H_2(W; \mathbb{Z}) = H_2(\pi; \mathbb{Z})$. The evaluation in question $H^2(\pi; K^*) \times H_2(\pi; \mathbb{Z}) \to K$ is induced by the bilinear form $K^* \times \mathbb{Z} \to K^*$, $(k, n) \mapsto k^n$, where $k \in K^*$ and $n \in \mathbb{Z}$. The summation on the right-hand side of (3.5.a) is the summation in K.

Proof. As we know, L splits as a direct sum of simple crossed Frobenius G-algebras $L^{(m)} = mL$ numerated by the orbits m of the action of G on I. Each $L^{(m)}$ is isomorphic to the crossed Frobenius G-algebra L^{G,H_m,θ_m,k_m}, where H_m is a subgroup of G of finite index, $\theta_m \in H^2(H_m; K^*)$, and $k_m \in K^*$. By Lemma 1.7.1 (a) and Corollaries 2.2 (b) and 3.2, the X-HQFT (A, τ) is isomorphic to the direct sum of the rescaled cohomological X-HQFTs $(A_m, \tau_m) = (A^{G,H_m,-\theta_m}, \tau^{G,H_m,-\theta_m,k_m})$. Therefore $\tau(W, \tilde{g}) = \sum_m \tau_m(W, \tilde{g})$.

For an orbit $m \subset I$ with two or more elements, $\tau_m(W, \tilde{g}) = 0$. Indeed, the HQFT (A_m, τ_m) is obtained by rescaling and transfer from the $K(H_m, 1)$-HQFT $(A^{-\theta_m}, \tau^{-\theta_m})$. Since $[G \colon H_m] = \mathrm{card}(m) \geq 2$, the covering $K(H_m, 1)$ of X is non-trivial. The assumption $g(\pi) = G$ implies that the map $\tilde{g} \colon W \to X$ does not lift to this covering. Therefore $\tau_m(W, \tilde{g}) = 0$.

A one-element orbit $m \subset I$ is just $\{i\}$ for $i \in I_0$. Then

$$H_m = G, \quad \theta_m = \nabla_i \in H^2(G; K^*), \quad k_m = F_L(i),$$

and (A_m, τ_m) is obtained from the primitive cohomological X-HQFT $(A^{-\theta_m}, \tau^{-\theta_m}) = (A^{-\nabla_i}, \tau^{-\nabla_i})$ by k_m-rescaling. Therefore

$$\tau_m(W, \tilde{g}) = \tau_{\{i\}}(W, \tilde{g}) = F_L(i)^{\chi(W)/2}\, g^*(-\nabla_i)([W]).$$

Hence

$$\tau(W,\tilde{g}) = \sum_{i\in I_0} \tau_{\{i\}}(W,\tilde{g}) = \sum_{i\in I_0} F_L(i)^{\chi(W)/2}\, g^*(-\nabla_i)([W]). \qquad \square$$

3.6 Remark. Theorem 3.5.1 can be generalized to a map $\tilde{g}\colon (W,w) \to (X,x)$ inducing an arbitrary, not necessarily surjective homomorphism $g\colon \pi \to G = \pi_1(X,x)$. Consider the covering $p\colon X' \to X$ determined by the subgroup $G' = g(\pi)$ of G. Consider the X'-HQFT (A',τ') obtained by pulling back the X-HQFT (A,τ) along p. It is clear that $\tilde{g}$ lifts to a map $\tilde{g}'\colon W \to X'$ inducing a surjective homomorphism of the fundamental groups. By the definition of τ', we have $\tau(W,\tilde{g}) = \tau'(W,\tilde{g}')$. If the crossed Frobenius G'-algebra L' underlying (A',τ') is semisimple, then Theorem 3.5.1 computes $\tau'(W,\tilde{g}')$ from the basic triple of L'. Note that L' is the restriction to G' of the crossed Frobenius G-algebra L underlying (A,τ). Clearly, L' is semisimple if and only if L is semisimple.

III.4 Proof of Theorem 3.1

4.1 The bijectivity for morphisms. We verify that for any two-dimensional X-HQFTs (A,τ), (A',τ') with underlying crossed Frobenius G-algebras L, L', the map

$$\mathrm{Iso}((A,\tau),(A',\tau')) \to \mathrm{Iso}(L,L') \tag{4.1.a}$$

constructed in Section 1.8 is bijective. The injectivity of this map is obvious since all X-curves are disjoint unions of loops and therefore any two isomorphisms $(A,\tau) \to (A',\tau')$ coinciding on loops coincide on all X-curves.

To establish the surjectivity of (4.1.a), we first show how to reconstruct (A,τ) from the underlying crossed Frobenius G-algebra $L = (L,\eta,\varphi)$. It is clear from the discussion in Section 1.1 that A is determined by L. We need only to reconstruct τ. It follows from the topological classification of compact oriented surfaces that every such surface splits along a finite family of disjoint simple loops into a disjoint union of disks, annuli, and disks with two holes. Therefore every X-surface W can be obtained by gluing from a finite family of X-disks with ≤ 2 holes. Using the axioms of an HQFT and Lemmas I.1.3.1–I.1.3.3, one easily deduces that $\tau(W)$ is determined by the values of τ on the X-disks with ≤ 2 holes. It remains to compute these values from (L,η,φ).

We begin by computing τ for annuli. Any X-annulus is X-homeomorphic to $C_{-+}(\alpha;\beta)$, or to $C_{--}(\alpha;\beta)$, or to $C_{++}(\alpha;\beta)$ with $\alpha,\beta \in G$. (Note that $C_{+-}(\alpha;\beta)$ is X-homeomorphic to $C_{-+}(\beta^{-1}\alpha\beta;\beta^{-1})$.) By definition, for $a \in L_\alpha$,

$$\tau(C_{-+}(\alpha;\beta))(a) = \varphi_{\beta^{-1}}(a). \tag{4.1.b}$$

Now the annulus $C_{--}(\alpha;\beta)$ can be obtained by gluing the annuli $C_{-+}(\alpha;\beta)$ and $C_{--}(\beta^{-1}\alpha\beta;1)$ along an X-homeomorphism $(C^1_+,\beta^{-1}\alpha\beta)\approx(C^0_-,\beta^{-1}\alpha\beta)$; see Figure III.7. The gluing axiom (1.2.6) and the definition of η imply that

$$\tau(C_{--}(\alpha;\beta))\in \mathrm{Hom}_K(L_\alpha\otimes L_{\beta^{-1}\alpha^{-1}\beta},K)$$

is computed by

$$\tau(C_{--}(\alpha;\beta))(a\otimes b)=\eta(\varphi_{\beta^{-1}}(a),b) \tag{4.1.c}$$

for any $a\in L_\alpha$ and $b\in L_{\beta^{-1}\alpha^{-1}\beta}$.

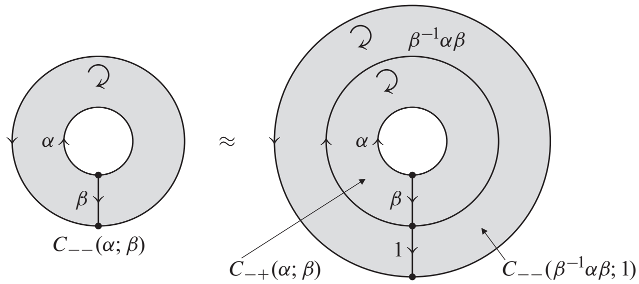


Figure III.7. A splitting of $C_{--}(\alpha;\beta)$.

To compute the vector

$$\tau(C_{++}(\alpha;\beta))\in \mathrm{Hom}_K(K,L_\alpha\otimes L_{\beta^{-1}\alpha^{-1}\beta})=L_\alpha\otimes L_{\beta^{-1}\alpha^{-1}\beta},$$

we expand it as a finite sum $\sum_i a_i\otimes b_i$, where $a_i\in L_\alpha$ and $b_i\in L_{\beta^{-1}\alpha^{-1}\beta}$. The gluing of $C_{--}(\alpha^{-1};1)$ to $C_{++}(\alpha;\beta)$ along an X-homeomorphism $(C^1_-,\alpha)\approx(C^0_+,\alpha)$ yields $C_{-+}(\alpha^{-1};\beta)$; see Figure III.8. The axioms of an HQFT and (4.1.b) yield that for any $a\in L_{\alpha^{-1}}$,

$$\sum_i \eta(a,a_i)\,b_i=\varphi_{\beta^{-1}}(a). \tag{4.1.d}$$

Since the restriction of η to $L_{\alpha^{-1}}\otimes L_\alpha$ is non-degenerate, (4.1.d) determines the vector $\tau(C_{++}(\alpha;\beta))=\sum_i a_i\otimes b_i$ uniquely.

There are two X-disks: B_+ where the orientation of the boundary is induced by that of the disk and B_- where the orientation of the boundary is opposite to the one induced from the disk; see Figure III.9. Then $\tau(B_+)=1_L\in L_1$. The X-disk B_- may be obtained by gluing B_+ and $C_{--}(1;1)$ along $\partial B_+\approx C^0_-$. Therefore $\tau(B_-)\in\mathrm{Hom}_K(L_1,K)$ is determined by L. More precisely, for $a\in L_1$,

$$\tau(B_-)(a)=\eta(1_L,a)=\eta(a,1_L). \tag{4.1.e}$$

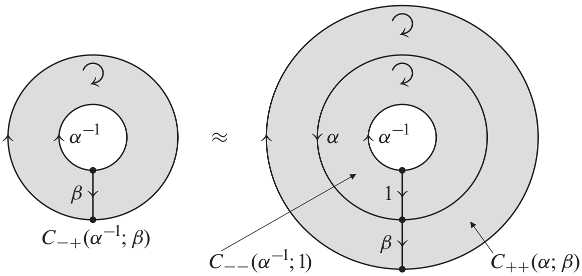


Figure III.8. A splitting of $C_{-+}(\alpha^{-1};\beta)$.

Figure III.9. The X-disks B_+ and B_-.

Each X-disk with two holes D splits along three disjoint simple loops parallel to the components of ∂D into a union of three X-annuli and a smaller X-disk with two holes. Choosing appropriate orientations of these three loops we can split D as a union of three X-annuli and an X-disk with two holes X-homeomorphic to $D_{--+}(\alpha,\beta;1,1)$ for some $\alpha,\beta\in G$. The homomorphism $\tau(D_{--+}(\alpha,\beta;1,1))$ is multiplication in L. The values of τ on the X-annuli were computed above. The axioms of an HQFT allow us to recover $\tau(D)$. We conclude that the HQFT (A,τ) can be entirely reconstructed from (L,η,φ).

We now prove the surjectivity of the map (4.1.a). Every isomorphism of crossed Frobenius G-algebras $\rho\colon L\to L'$ defines in the obvious way a K-isomorphism $\rho_M: A_M\to A'_M$ for any X-curve $M\approx S^1$. These isomorphisms extend to arbitrary X-curves M by multiplicativity. We claim that the resulting family of isomorphisms $\{\rho_M: A_M\to A'_M\}_M$ is a morphism of HQFTs. To see this, we must check that these isomorphisms make the natural square diagrams associated with X-homeomorphisms and X-surfaces commutative. The part concerning homeomorphisms is obvious; see the discussion in Section 1.1. As explained above, every X-surface can be glued from a finite family of X-surfaces of types $B_+, C_{\varepsilon,\mu}(\alpha;\beta)$, and $D_{--+}(\alpha,\beta;1,1)$, where $\alpha,\beta\in G$ and $\varepsilon,\mu=\pm$. Therefore it suffices to check the commutativity of the square diagrams associated with these X-surfaces. For B_+ and $D_{--+}(\alpha,\beta;1,1)$, the required commutativity follows from the assumption that $\rho\colon L\to L'$ is an algebra isomorphism. For the X-annuli, the commutativity follows from the formulas (4.1.b)–(4.1.d) since ρ preserves the inner product and commutes with the action of G on L and L'.

4.2 The surjectivity for objects. To complete the proof of Theorem 3.1, it remains to show that any crossed Frobenius G-algebra $(L = \bigoplus_{\alpha\in G} L_\alpha, \eta, \varphi)$ can be realized as the underlying G-algebra of a two-dimensional X-HQFT (A, τ).

We associate with every X-curve M a K-module A_M as follows. If $M = \emptyset$, then $A_M = K$. If $M \approx S^1$ and M represents $\alpha \in G$, then $A_M = L_\alpha$. If M has two or more components $\{M_i\}_i$, then $A_M = \otimes_i A_{M_i}$.

For an arbitrary X-homeomorphism $f: M \to M'$, the induced isomorphism $f_\#: A_M \to A_{M'}$ is the identity map $\mathrm{id}: K \to K$ if $M = M' = \emptyset$ and the identity map $\mathrm{id}: L_\alpha \to L_\alpha$ if $M \approx S^1 \approx M'$ and M, M' represent $\alpha \in G$. For non-connected M, M', the isomorphism $f_\#: A_M \to A_{M'}$ is the obvious tensor product of the identity homomorphisms.

The construction of τ proceeds in seven steps.

Step 1. We define τ for X-annuli using (4.1.b)–(4.1.d). To establish the topological invariance of τ (Axiom (1.2.4)) note that the X-homeomorphisms of X-annuli are generated (up to isotopy) by

(i) the Dehn twists $C_{\varepsilon,\mu}(\alpha;\beta) \to C_{\varepsilon,\mu}(\alpha;\alpha\beta)$ about the circle $S^1 \times (1/2) \subset C = S^1 \times [0,1]$, where $\varepsilon, \mu = \pm$, and

(ii) the reflection $C_{\varepsilon,\varepsilon}(\alpha;\beta) \to C_{\varepsilon,\varepsilon}(\beta^{-1}\alpha^{-1}\beta;\beta^{-1})$ permuting the boundary components of C and defined as the product of the orientation reversing involutions $S^1 \to S^1$, $s \mapsto -\bar{s}$ and $[0,1] \to [0,1]$, $t \mapsto 1-t$.

The invariance of τ under the Dehn twists (i) follows from (4.1.b)–(4.1.d) since

$$\varphi_{(\alpha\beta)^{-1}}|_{L_{\alpha^{\pm 1}}} = \varphi_{\beta^{-1}\alpha^{-1}}|_{L_{\alpha^{\pm 1}}} = \varphi_{\beta^{-1}}\,\varphi_{\alpha^{-1}}|_{L_{\alpha^{\pm 1}}} = \varphi_{\beta^{-1}}|_{L_{\alpha^{\pm 1}}}.$$

The invariance of $\tau(C_{--}(\alpha;\beta)): L_\alpha \otimes L_{\beta^{-1}\alpha^{-1}\beta} \to K$ under the reflection (ii) follows from (4.1.c) since for any $a \in L_\alpha$ and $b \in L_{\beta^{-1}\alpha^{-1}\beta}$,

$$\begin{aligned}\tau(C_{--}(\alpha;\beta))(a\otimes b) &= \eta(\varphi_{\beta^{-1}}(a), b)\\ &= \eta(a, \varphi_\beta(b))\\ &= \eta(\varphi_\beta(b), a) = \tau(C_{--}(\beta^{-1}\alpha^{-1}\beta;\beta^{-1}))(b\otimes a).\end{aligned}$$

To show the invariance of $\tau(C_{++}(\alpha;\beta)) = \sum_i a_i \otimes b_i \in L_\alpha \otimes L_{\beta^{-1}\alpha^{-1}\beta}$ under the reflection (ii), it suffices to deduce from (4.1.d) that for any $b \in L_{\beta^{-1}\alpha\beta}$,

$$\sum_i \eta(b, b_i)\, a_i = \varphi_\beta(b). \tag{4.2.a}$$

Given $a \in L_{\alpha^{-1}}$,

$$\begin{aligned}\eta(a, \sum_i \eta(b,b_i)\,a_i) &= \sum_i \eta(a,a_i)\,\eta(b,b_i)\\ &= \eta(b, \sum_i \eta(a,a_i)\,b_i)\\ &= \eta(b, \varphi_{\beta^{-1}}(a)) = \eta(\varphi_{\beta^{-1}}(a), b) = \eta(a, \varphi_\beta(b)).\end{aligned}$$

Now, the non-degeneracy of η implies (4.2.a).

Step 2. Consider the gluing of the X-annuli $C_{\varepsilon,\mu}(\alpha;\beta)$ and $C_{-\mu,\nu}(\gamma;\delta)$ along an X-homeomorphism $(C^1_\mu, \beta^{-1}\alpha^{-\varepsilon\mu}\beta) \approx (C^0_{-\mu}, \gamma)$, where $\alpha,\beta,\gamma,\delta \in G$ and $\varepsilon,\mu,\nu = \pm$ are such that $\gamma = \beta^{-1}\alpha^{-\varepsilon\mu}\beta$. The gluing yields the X-annulus $C_{\varepsilon,\nu}(\alpha;\beta\delta)$. We verify here that the vectors τ associated with these three X-annuli are related by the gluing rule formulated at the end of Section 1.1. Note that the gluing of $C_{\varepsilon,-}$ to $C_{+,\nu}$ along $C^1_- \approx C^0_+$ is topologically equivalent to the gluing of $C_{\nu,+}$ to $C_{-,\varepsilon}$ along $C^1_+ \approx C^0_-$. Therefore without loss of generality we can assume that $\mu = +$. We indicate now the key algebraic argument behind the gluing rule; the detailed computations are left to the reader.

Case (1): $\varepsilon = -$, $\nu = +$. Use the identity $\varphi_{\delta^{-1}}\varphi_{\beta^{-1}} = \varphi_{(\beta\delta)^{-1}}$.

Case (2): $\varepsilon = \nu = -$. Use again that $\varphi_{\delta^{-1}}\varphi_{\beta^{-1}} = \varphi_{(\beta\delta)^{-1}}$.

Case (3): $\varepsilon = \nu = +$. The required formula is equivalent to the identity $\sum_i \eta(a,a_i)\,\varphi_{\delta^{-1}}(b_i) = \varphi_{(\beta\delta)^{-1}}(a)$ for any $a \in L_{\alpha^{-1}}$. This identity follows from (4.1.d).

Case (4): $\varepsilon = +$, $\nu = -$. The required formula is equivalent to the following equality of homomorphisms from $L_{(\beta\delta)^{-1}\alpha\beta\delta} = L_{\delta^{-1}\gamma^{-1}\delta}$ to L_α:

$$(\mathrm{id}_{L_\alpha} \otimes \tau(C_{--}(\gamma;\delta))) \circ (\tau(C_{++}(\alpha;\beta)) \otimes \mathrm{id}_{L_{(\beta\delta)^{-1}\alpha\beta\delta}}) = \tau(C_{+-}(\alpha;\beta\delta)).$$

The right-hand side carries any $a \in L_{(\beta\delta)^{-1}\alpha\beta\delta}$ to $\varphi_{\beta\delta}(a)$ by the definition of $\varphi_{\beta\delta}$. Using an expansion $\tau(C_{++}(\alpha;\beta)) = \sum_i a_i \otimes b_i \in L_\alpha \otimes L_\gamma$ as above and formulas (4.1.c), (4.2.a). we compute that the left-hand side carries $a \in L_{(\beta\delta)^{-1}\alpha\beta\delta}$ to

$$\begin{aligned}(\mathrm{id}_{L_\alpha} \otimes \tau(C_{--}(\gamma;\delta)))\Big(\sum_i a_i \otimes b_i \otimes a\Big) &= \sum_i a_i\,\eta(\varphi_{\delta^{-1}}(b_i), a)\\ &= \sum_i a_i\,\eta(b_i, \varphi_\delta(a))\\ &= \varphi_\beta\varphi_\delta(a) = \varphi_{\beta\delta}(a).\end{aligned}$$

Step 3. At Step 3 we define τ for X-disks with two holes. For an X-disk with two holes $D = D_{--+}(\alpha,\beta;\rho,\delta)$ (see Section 1.2 for notation), we define $\tau(D) \in \mathrm{Hom}_K(L_\alpha \otimes L_\beta, L_{\rho\alpha\rho^{-1}\delta\beta\delta^{-1}})$ by $\tau(D)(a \otimes b) = \varphi_\rho(a)\,\varphi_\delta(b)$, where $a \in L_\alpha$ and $b \in L_\beta$. Observe that the group of isotopy classes of orientation preserving self-homeomorphisms of D_{--+} is generated by Dehn twists in cylinder neighborhoods of the circles $Y, Z \subset \partial D_{--+}$ and the homeomorphism $f\colon D_{--+} \to D_{--+}$ introduced in the proof of (3.1.2) in Lemma 1.6. That τ is preserved under the Dehn twists is proven by the same computation as the topological invariance of τ for X-annuli at Step 1. That τ is preserved under f means that the homomorphisms

$$\tau(D_{--+}(\alpha,\beta;1,1))\colon L_\alpha \otimes L_\beta \to L_{\alpha\beta}$$

and

$$\tau(D_{--+}(\beta,\alpha;1,\beta^{-1}))\colon L_\beta \otimes L_\alpha \to L_{\alpha\beta}$$

are obtained from each other by the flip $L_\alpha \otimes L_\beta \to L_\beta \otimes L_\alpha$. This follows from Axiom (3.1.2) using essentially the same computation as in the proof of this axiom in Lemma 1.6.

Consider now an X-disk with two holes $D = D_{---}(\alpha, \beta; \rho, \delta)$ and set $\gamma = \delta\beta^{-1}\delta^{-1}\rho\alpha^{-1}\rho^{-1} \in G$. We define $\tau(D) \in \mathrm{Hom}_K(L_\alpha \otimes L_\beta \otimes L_\gamma, K)$ by

$$\tau(D)(a \otimes b \otimes c) = \eta(\varphi_\rho(a)\,\varphi_\delta(b), c),$$

where $a \in L_\alpha$, $b \in L_\beta$, $c \in L_\gamma$. This definition results immediately from the gluing rule if we present D as the result of a gluing of $D_{--+}(\alpha, \beta; \rho, \delta)$ and $C_{--}(\gamma^{-1}; 1)$ along $(T_+, \gamma^{-1}) \approx (C_-^0, \gamma^{-1})$. This definition ensures the gluing rule for this gluing.

To verify the topological invariance, consider an orientation preserving homeomorphism $h\colon D_{---} \to D_{---}$ which carries (Y, y), (Z, z), (T, t) onto (Z, z), (T, t), (Y, y), respectively. We choose h so that the arc tz is mapped onto $yt = (ty)^{-1}$ and the arc ty is mapped onto an embedded arc leading from y to z and homotopic to the product of the arcs yt and tz. An easy computation shows that h is an X-homeomorphism $D_{---}(\alpha, \beta; \rho, \delta) \to D_{---}(\gamma, \alpha; \delta^{-1}, \delta^{-1}\rho)$. It is clear that any orientation preserving self-homeomorphism of D_{---} (considered up to isotopy) expands as a composition of Dehn twists in cylinder neighborhoods of Y, Z, T and, eventually, $h^{\pm 1}$. The invariance of $\tau(D)$ under the Dehn twists in questions follows from the already established topological invariance of $\tau(D_{--+}(\alpha, \beta; \rho, \delta))$ and $\tau(C_{--}(\gamma^{-1}; 1))$. The invariance of τ under h follows from the equalities

$$\begin{aligned}\eta(\varphi_\rho(a)\,\varphi_\delta(b), c) &= \eta(c, \varphi_\rho(a)\,\varphi_\delta(b))\\ &= \eta(c\,\varphi_\rho(a), \varphi_\delta(b))\\ &= \eta(\varphi_{\delta^{-1}}(c)\,\varphi_{\delta^{-1}\rho}(a), b).\end{aligned}$$

Now we define τ for $D = D_{++-}(\alpha, \beta; \rho, \delta)$. We can obtain D by gluing three X-annuli $C_{++}(\alpha; 1)$, $C_{++}(\beta; 1)$ and $C_{--}(\gamma; 1)$, where $\gamma = \rho\alpha^{-1}\rho^{-1}\delta\beta^{-1}\delta^{-1}$, to $D_{--+}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$ along the X-homeomorphisms, respectively,

$$(C_+^1, \alpha^{-1}) \approx (Y_-, \alpha^{-1}), \quad (C_+^1, \beta^{-1}) \approx (Z_-, \beta^{-1}), \quad (C_-^0, \gamma) \approx (T_+, \gamma).$$

The gluing rule determines $\tau(D)$ uniquely. The topological invariance of $\tau(D)$ follows from the topological invariance of the values of τ on $D_{--+}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$ and on the three X-annuli above since any self-homeomorphism of D is isotopic to a homeomorphism preserving the union of these annuli set-wise.

Similarly, we can obtain $D_{+++}(\alpha, \beta; \rho, \delta)$ by gluing three X-annuli of type $C_{++}(\cdot; 1)$ to the X-disk with two holes $D_{---}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$. The gluing rule determines $\tau(D_{+++}(\alpha, \beta; \rho, \delta))$ in a topologically invariant way.

Note finally that any X-disk with two holes D' is X-homeomorphic to one of the X-disks with two holes D considered above. This determines $\tau(D')$ uniquely. Different choices of D and of the X-homeomorphism $D \approx D'$ lead to the same $\tau(D')$ because of the topological invariance of $\tau(D)$ established above.

Step 4. At Step 4 we check the gluing rule for gluings of an X-annulus to an X-disk with two holes along one boundary circle. Consider first a gluing of C_{-+} to D_{--+} (this again produces D_{--+}). If the gluing is performed along an X-homeomorphism $C_+^1 \approx Y_-$ or $C_+^1 \approx Z_-$, then the gluing rule follows from the identity $\varphi_\rho \varphi_{\rho'} = \varphi_{\rho\rho'}$. If the gluing is performed along $C_-^0 \approx T_+$, then the gluing rule follows from the assumption that all φ_ρ are algebra homomorphisms.

Consider next a gluing of C_{-+} to D_{---} (this produces D_{---}). If the gluing is performed along Y or Z, then the gluing rule follows from the identity $\varphi_\alpha \varphi_\beta = \varphi_{\alpha\beta}$. The existence of a self-homeomorphism of D_{---} mapping T onto Y shows that the gluing rule holds also for the gluings along T.

Consider now an arbitrary gluing of an X-annulus $C_{\varepsilon^0,\varepsilon^1}$ to an X-disk with two holes $D_{\varepsilon,\mu,\nu}$ along one boundary circle. By the topological invariance of τ established at Steps 1 and 3, it is sufficient to consider the gluing performed along an X-homeomorphism $C^0_{\varepsilon^0} \approx T_\nu$ so that $\varepsilon^0 = -\nu$. This gluing yields $D_{\varepsilon,\mu,\varepsilon^1}$. There are 16 cases corresponding to different signs $\varepsilon^1, \varepsilon, \mu, \nu$. The cases where $\varepsilon^1 = \nu (= -\varepsilon^0)$ and the triple ε, μ, ν contains at least 2 minuses were considered above at this step. We check the remaining cases one by one using the fact that tensor contractions along different tensor factors commute. We begin with the case $\varepsilon = \mu$ and indicate the key argument leaving the details to the reader.

Case $\varepsilon = \mu = -$, $\nu = +$, $\varepsilon^1 = -$. The gluing rule follows from the definition of $\tau(D_{---})$ and the gluing rule for annuli obtained on Step 2.

Case $\varepsilon = \mu = -$, $\nu = -$, $\varepsilon^1 = +$. The gluing rule follows from the definition of $\tau(D_{---})$ and the gluing rule for annuli.

Case $\varepsilon = \mu = +$, $\nu = +$, $\varepsilon^1 = +$. The gluing rule follows from the definition of $\tau(D_{+++})$ and the gluing rule for annuli.

Case $\varepsilon = \mu = +$, $\nu = -$, $\varepsilon^1 = +$. The gluing rule follows from the definitions of $\tau(D_{++-})$, $\tau(D_{+++})$ and the gluing rule for annuli.

Case $\varepsilon = \mu = +$, $\nu = -$, $\varepsilon^1 = -$. The gluing rule follows from the definition of $\tau(D_{++-})$ and the gluing rule for annuli.

Case $\varepsilon = \mu = +$, $\nu = +$, $\varepsilon^1 = -$. The gluing rule follows from the definitions of $\tau(D_{+++})$, $\tau(D_{++-})$ and the gluing rule for annuli.

The case $\varepsilon = -$, $\mu = +$ reduces to $\varepsilon = +$, $\mu = -$ by the topological invariance (Step 3). Assume that $\varepsilon = +$ and $\mu = -$. If $\nu = \varepsilon^1 = +$, then the gluing rule follows from the definition of $\tau(D_{+-+})$. We are left with two cases ($\nu = +$, $\varepsilon^1 = -$) and ($\nu = -$, $\varepsilon^1 = +$). The gluings in question $D_{+-+} \mapsto D_{+--}$ and $D_{+--} \mapsto D_{+-+}$ are mutually inverse and therefore (taking into account the results obtained at Step 2) it is enough to consider one of them. We consider the first gluing in detail.

Let $\alpha, \beta, \rho, \delta, \zeta \in G$ and $\gamma = \rho\alpha^{-1}\rho^{-1}\delta\beta\delta^{-1} \in G$. Gluing the X-annulus $C_{--}(\gamma;\zeta)$ to $D_1 = D_{+-+}(\alpha,\beta;\rho,\delta)$ along $(C_-^0, \gamma) \approx (T_+, \gamma)$ we obtain the X-disk with two holes

$$D_2 = D_{+--}(\alpha, \beta; \zeta^{-1}\rho, \zeta^{-1}\delta).$$

Clearly, D_2 is X-homeomorphic to

$$D_{--+}(\beta, \zeta^{-1}\gamma^{-1}\zeta; \rho^{-1}\delta, \rho^{-1}\zeta).$$

By Step 3, the map $\tau(D_2)\colon L_\beta \otimes L_{\zeta^{-1}\gamma^{-1}\zeta} \to L_\alpha$ carries $a \otimes b$ with $a \in L_\beta$, $b \in L_{\zeta^{-1}\gamma^{-1}\zeta}$ to $\varphi_{\rho^{-1}\delta}(a)\, \varphi_{\rho^{-1}\zeta}(b)$. To compute $\tau(D_1)$ following the instructions given at Step 3, present D_1 as the result of the gluing of the X-annuli $C_{++}(\alpha; 1)$, $C_{--}(\beta; 1)$, $C_{++}(\gamma^{-1}; 1)$ to $D_3 = D_{-+-}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$ along X-homeomorphisms, respectively,

$$(C_+^1, \alpha^{-1}) \approx (Y_-, \alpha^{-1}), \quad (C_-^1, \beta^{-1}) \approx (Z_+, \beta^{-1}), \quad (C_+^0, \gamma^{-1}) \approx (T_-, \gamma^{-1}).$$

The vector $\tau(D_1)$ is obtained by the tensor contraction from the tensor product of four τ-vectors corresponding to these four pieces. We must prove that $\tau(D_2) = \tau'$, where τ' is the result of the tensor contraction of $\tau(D_1) \otimes \tau(C_{--}(\gamma; \zeta))$ along the two dual tensor factors contributed by $(C_-^0, \gamma) \approx (T_+, \gamma)$. The gluing of $C_{++}(\gamma^{-1}; 1)$ to $C_{--}(\gamma; \zeta)$ along $(C_+^1, \gamma) \approx (C_-^0, \gamma)$ gives $C_{+-}(\gamma^{-1}; \zeta)$. By Step 2, the tensor contraction of $\tau(C_{++}(\gamma^{-1}; 1)) \otimes \tau(C_{--}(\gamma; \zeta))$ along the two dual tensor factors contributed by $(C_+^1, \gamma) \approx (C_-^0, \gamma)$ is equal to $\tau(C_{+-}(\gamma^{-1}; \zeta))$. The gluing of $C_{+-}(\gamma^{-1}; \zeta)$ to D_3 along $(C_+^0, \gamma^{-1}) \approx (T_-, \gamma^{-1})$ gives

$$D_4 = D_{-+-}(\alpha^{-1}, \beta^{-1}; \zeta^{-1}\rho, \zeta^{-1}\delta).$$

By Step 3, the tensor contraction of $\tau(C_{+-}(\gamma^{-1}; \zeta)) \otimes \tau(D_3)$ along the two dual tensor factors contributed by $(C_+^0, \gamma^{-1}) = (T_-, \gamma^{-1})$ is equal to $\tau(D_4)$. Therefore, τ' is obtained from

$$\tau(C_{++}(\alpha; 1)) \otimes \tau(C_{--}(\beta; 1)) \otimes \tau(D_4)$$

by the tensor contraction along the dual tensor factors contributed by the circles $(C_+^1, \alpha^{-1}) \approx (Y_-, \alpha^{-1})$ and $(C_-^1, \beta^{-1}) \approx (Z_+, \beta^{-1})$. Expand

$$\tau(C_{++}(\alpha; 1)) = \sum_i a_i \otimes b_i$$

with $a_i \in L_\alpha$ and $b_i \in L_{\alpha^{-1}}$. It is easy to see that D_4 is X-homeomorphic to $D_{--+}(\zeta^{-1}\gamma\zeta, \alpha^{-1}; \delta^{-1}\zeta, \delta^{-1}\rho)$. This implies that for any $a \in L_\beta, b \in L_{\zeta^{-1}\gamma^{-1}\zeta}$, the value of $\tau' \in \operatorname{Hom}(L_\beta \otimes L_{\zeta^{-1}\gamma^{-1}\zeta}, L_\alpha)$ on $a \otimes b$ is computed by

$$\begin{aligned}
\tau'(a \otimes b) &= \sum_i \eta(\varphi_{\delta^{-1}\zeta}(b)\varphi_{\delta^{-1}\rho}(b_i), a)\, a_i \\
&= \sum_i \eta(\varphi_\zeta(b)\varphi_\rho(b_i), \varphi_\delta(a))\, a_i \\
&= \sum_i \eta(\varphi_\delta(a)\varphi_\zeta(b), \varphi_\rho(b_i))\, a_i \\
&= \sum_i \eta(\varphi_{\rho^{-1}\delta}(a)\varphi_{\rho^{-1}\zeta}(b), b_i)\, a_i \\
&= \varphi_{\rho^{-1}\delta}(a)\, \varphi_{\rho^{-1}\zeta}(b),
\end{aligned}$$

where the last equality follows from the definition of $\tau(C_{++}(\alpha; 1))$. We conclude that $\tau' = \tau(D_2)$.

Step 5. Now we define τ for the X-disks B_+ and B_- (see Section 4.1 for notation). Set $\tau(B_+) = 1_L \in L_1$. The gluing rule for a gluing of B_+ to an X-annulus of type C_{-+} follows from the equality $\varphi_\beta(1_L) = 1_L$ for $\beta \in G$. The gluing rule for a gluing of B_+ to an X-disk with two holes of type D_{--+} follows from the equalities $1_L a = a 1_L = a$ for $a \in L$. This fact and the definition of τ for X-disks with two holes of types D_{---} or D_{++-} imply the gluing rule for any gluing of B_+ to such disks with holes.

The disk B_- can be obtained by gluing B_+ and $C_{--}(1; 1)$ along $\partial B_+ \approx C_-^0$. This determines $\tau(B_-) \in \mathrm{Hom}_K(L_1, K)$. The gluing rule for gluings of the disk B_- to X-disks with ≤ 2 holes follows from the already established properties of the gluings of $C_{--}(1; 1)$ and B_+.

Step 6. Now we define $\tau(W)$ for any connected X-surface W. By a *splitting system of loops* on W we mean a finite family of disjoint embedded circles $\alpha_1, \dots, \alpha_N \subset W$ which split W into a union of disks with ≤ 2 holes. We provide each α_i with an orientation and a base point x_i. We choose a map $g\colon W \to X$ in the given homotopy class so that $g(x_i) = x \in X$ for all i. The disks with holes obtained by splitting W along $\bigcup_i \alpha_i$ endowed with the restrictions of g are X-surfaces. The gluing rule determines $\tau(W)$ from the values of τ on these X-disks with holes.

We claim that $\tau(W)$ does not depend on the choice of orientations and base points on $\alpha_1, \dots, \alpha_N$. The proof is as follows. Pick $i \in \{1, \dots, N\}$ and set $\alpha = \alpha_i$. Let D_1, D_2 be the disks with holes adjoint to α from different sides (they may coincide). A cylinder neighborhood C of α in D_1 is an annulus bounded by α and a parallel loop $\tilde{\alpha} \subset \mathrm{Int}\, D_1$. Consider the disks with holes $\widetilde{D}_1 = \overline{D_1 - C}$ and $\widetilde{D}_2 = \overline{D_2 \cup C}$. We provide $\tilde{\alpha}$ with a base point $\tilde{x}$ and the orientation opposite to that of α. We deform g in a small neighborhood of $\tilde{x}$ so that $g(\tilde{x}) = x$. In this way the surfaces $C, \widetilde{D}_1, \widetilde{D}_2$ become X-surfaces. It follows from the properties of τ established above that

$$\tau(D_1) = *_{\tilde{\alpha}}(\tau(\widetilde{D}_1) \otimes \tau(C)) \quad \text{and} \quad \tau(\widetilde{D}_2) = *_\alpha(\tau(C) \otimes \tau(D_2)),$$

where $*_\alpha$ denotes the tensor contraction corresponding to the gluing along α. Then

$$*_\alpha(\tau(D_1) \otimes \tau(D_2)) = *_\alpha *_{\tilde{\alpha}} (\tau(\widetilde{D}_1) \otimes \tau(C) \otimes \tau(D_2)) = *_{\tilde{\alpha}}(\tau(\widetilde{D}_1) \otimes \tau(\widetilde{D}_2)).$$

Thus, replacing α with $\tilde{\alpha}$ in the splitting system of loops we do not change $\tau(W)$. This implies that $\tau(W)$ does not depend on the choice of orientations and base points on $\alpha_1, \dots, \alpha_N$. A similar argument shows that $\tau(W)$ does not depend on the choice of g in its homotopy class (relative to the base points on ∂W).

We claim that the homomorphism $\tau(W)$ does not depend on the choice of a splitting system of loops on W. The crucial argument is provided by the theorem, due to A. Hatcher and W. Thurston [HT], that any two splitting systems of loops on W are related by a finite sequence of the following transformations on splitting systems of loops on W:

(i) isotopy in W;
(ii) addition of an arbitrary simple loop disjoint from the given loops;
(iii) deletion of one loop from a splitting system of loops, provided the remaining loops form a splitting system;
(iv) replacement of a loop α adjacent to different disks with two holes D_1, D_2, by a simple loop in Int $D_1 \cup$ Int $D_2 \cup \alpha$, meeting α transversely in two points and splitting both D_1 and D_2 into annuli;
(v) replacement of a loop α in a splitting system by a simple loop meeting α transversely in one point and disjoint from the other loops.

We should check the invariance of $\tau(W)$ under these transformations. The invariance of $\tau(W)$ under isotopy is obvious. Consider the transformation (ii) adding a loop α in $W - \bigcup_i \alpha_i$. The loop α lies in a disk with ≤ 2 holes obtained by the splitting of W along $\bigcup_i \alpha_i$. The loop α splits this disk with ≤ 2 holes into a union of a smaller disk with ≤ 2 holes and an annulus or a disk without holes. The invariance of $\tau(W)$ follows from the already established gluing rule for gluings of a disk with ≤ 2 holes to an annulus or a disk along one boundary component. The transformation (iii) is inverse to (ii) and the same argument applies. To handle (iv), observe that the associativity of multiplication in L yields two equivalent expressions for the value of τ for a disk with three holes; these expressions are obtained from two splittings of the disk with three holes as a union of two disks with two holes; see the proof of Lemma 1.6. Since we are free to choose orientations of the loops in a splitting system, we can reduce the invariance of $\tau(W)$ under the transformation (iv) to this model case. Note that in order to have the maps to X as in the model case (i.e., mapping the arcs ty, tz to the base point $x \in X$) we can use transformations (ii) to add additional annuli to the splitting. Similarly, the invariance of $\tau(W)$ under the transformation (v) follows from the torus condition (Axiom (3.1.4)), cf. the proof of Axiom (3.1.4) in Lemma 1.6.

The topological invariance of $\tau(W)$ follows from the topological invariance of τ for disks with ≤ 2 holes and the fact that any homeomorphism of connected surfaces maps a splitting system of loops onto a splitting system of loops.

Step 7. We already defined τ for connected X-surfaces. We extend τ to arbitrary X-surfaces by the tensor multiplicativity (Axiom (1.2.5)). It follows from the definitions that (A, τ) is an X-HQFT. (To verify gluing axiom (1.2.6), we simply compute $\tau(W)$ using a splitting system of loops containing $N = N' \subset W$.) It is clear that (L, η, φ) is the underlying crossed Frobenius G-algebra of (A, τ).

III.5 Hermitian two-dimensional HQFTs

In this section we assume that the ground ring K is endowed with a ring involution $K \to K, k \mapsto \bar{k}$.

5.1 Theorem. *The underlying crossed Frobenius G-algebra of a Hermitian two-dimensional HQFT over K with target $X = K(G,1)$ has a natural Hermitian involution. This establishes an equivalence between the category $HQ_2(X)$ of two-dimensional Hermitian X-HQFTs over K and the category $\mathcal{HQ}(G)$ of Hermitian crossed Frobenius G-algebras over K.*

Proof. Consider a Hermitian two-dimensional X-HQFT (A,τ) and its underlying crossed Frobenius G-algebra L. The Hermitian structure on (A,τ) yields for each $\alpha \in G$ a non-degenerate Hermitian pairing $\langle\cdot\,,\,\cdot\rangle_\alpha\colon L_\alpha \times L_\alpha \to K$. By the non-degeneracy of η, there is a unique antilinear isomorphism $L_\alpha \to L_{\alpha^{-1}}$, $b \mapsto \bar{b}$ such that for any $a,b \in L_\alpha$,

$$\langle a,b\rangle_\alpha = \eta(a,\bar{b}). \tag{5.1.a}$$

The Hermitian symmetry $\langle a,b\rangle_\alpha = \overline{\langle b,a\rangle_\alpha}$ may be rewritten as

$$\eta(a,\bar{b}) = \overline{\eta(\bar{a},b)} \tag{5.1.b}$$

for any $a,b \in L$. We now check that the homomorphism $L \to L$, $b \mapsto \bar{b}$ is a Hermitian involution on L. To this end, we shall apply Axiom (4.1.2), p. 16, of a Hermitian X-HQFT to several X-surfaces.

For $W = C_{-+}(\alpha;\beta)$, we have $-W = C_{-+}(\beta^{-1}\alpha\beta;\beta^{-1})$. By definition,

$$\tau(W) = \varphi_{\beta^{-1}}\colon L_\alpha \to L_{\beta^{-1}\alpha\beta} \quad \text{and} \quad \tau(-W) = \varphi_\beta\colon L_{\beta^{-1}\alpha\beta} \to L_\alpha.$$

Axiom (4.1.2) for this W may be rewritten as the identity

$$\langle\varphi_{\beta^{-1}}(a), b\rangle_{\beta^{-1}\alpha\beta} = \langle a, \varphi_\beta(b)\rangle_\alpha$$

for any $a \in L_\alpha$ and $b \in L_{\beta^{-1}\alpha\beta}$. Therefore

$$\eta(\varphi_{\beta^{-1}}(a),\bar{b}) = \eta(a,\overline{\varphi_\beta(b)}) = \eta(\varphi_{\beta^{-1}}(a),\varphi_{\beta^{-1}}(\overline{\varphi_\beta(b)})).$$

Since η is non-degenerate, the latter formula implies that $\varphi_\beta(\bar{b}) = \overline{\varphi_\beta(b)}$.

For $W = C_{--}(\alpha;1)$, we have $-W = C_{++}(\alpha;1)$. By definition, $\tau(W) = \eta\colon L_\alpha \otimes L_{\alpha^{-1}} \to K$. The homomorphism $\tau(-W)\colon K \to L_\alpha \otimes L_{\alpha^{-1}}$ carries 1_K into a sum $\sum_i a_i \otimes b_i$, where $a_i \in L_\alpha, b_i \in L_{\alpha^{-1}}$, and $\sum_i \eta(a,a_i)\,b_i = a$ for any $a \in L_{\alpha^{-1}}$, cf. formula (4.1.d). Axiom (4.1.2) for this W and Axiom (4.1.1) imply that for any $c \in L_\alpha$ and $d \in L_{\alpha^{-1}}$,

$$\begin{aligned}
\eta(c,d) &= \langle\eta(c,d),1_K\rangle_\emptyset \\
&= \sum_i \langle c,a_i\rangle_\alpha \langle d,b_i\rangle_{\alpha^{-1}} \\
&= \sum_i \eta(c,\bar{a}_i)\,\eta(d,\bar{b}_i) \\
&= \overline{\sum_i \eta(\bar{c},a_i)\,\eta(\bar{d},b_i)} \\
&= \overline{\eta(\bar{d},\sum_i \eta(\bar{c},a_i)b_i)} = \overline{\eta(\bar{d},\bar{c})} = \overline{\eta(\bar{c},\bar{d})}.
\end{aligned}$$

This gives $\overline{\eta(c,d)} = \eta(\bar{c},\bar{d})$. Using this formula and the non-degeneracy of η, we obtain that (5.1.b) is equivalent to the involutivity of the map $L \to L, b \mapsto \bar{b}$.

For $W = D_{--+}(\alpha,\beta;1,1)$, the homomorphism $\tau(W)$ is the algebra multiplication $L_\alpha \otimes L_\beta \to L_{\alpha\beta}$. The X-surface $-W$ is X-homeomorphic to $D_{++-}(\beta,\alpha;1,1)$. The homomorphism $\tau(-W)\colon L_{\alpha\beta} \to L_\beta \otimes L_\alpha$ can be computed using the fact that the gluing of $C_{--}(\beta^{-1};1)$ to $D_{++-}(\beta,\alpha;1,1)$ along $(C^1_-,\beta) \approx (Y_+,\beta)$ gives an X-disk with two holes X-homeomorphic to $D_{--+}(\alpha\beta,\beta^{-1};1,1)$. This implies that $\tau(-W)$ carries any $c \in L_{\alpha\beta}$ to a sum $\sum_j b_j \otimes a_j$ with $b_j \in L_\beta, a_j \in L_\alpha$ such that $cd = \sum_j \eta(d,b_j)a_j$ for all $d \in L_{\beta^{-1}}$. Axiom (4.1.2) for W gives

$$\langle ab,c\rangle_{\alpha\beta} = \sum_j \langle a,a_j\rangle_\alpha \langle b,b_j\rangle_\beta$$

for any $a \in L_\alpha, b \in L_\beta, c \in L_{\alpha\beta}$. The left-hand side is equal to $\eta(ab,\bar{c}) = \eta(a,b\bar{c})$ while the right-hand side is equal to

$$\sum_j \eta(a,\bar{a}_j)\,\eta(b,\bar{b}_j) = \overline{\sum_j \eta(\bar{a},a_j)\,\eta(\bar{b},b_j)} = \overline{\eta(\bar{a},\sum_j \eta(\bar{b},b_j)a_j)}$$
$$= \overline{\eta(\bar{a},c\bar{b})} = \eta(a,\overline{c\bar{b}}).$$

By the non-degeneracy of η, we have $b\bar{c} = \overline{c\bar{b}}$ for any $b,c \in L$. Hence $\bar{b}\bar{c} = \overline{cb}$. We conclude that $b \mapsto \bar{b}$ is a Hermitian involution on L.

Conversely, having a Hermitian involution on L we define a Hermitian structure on (A,τ) using (5.1.a) and Axiom (4.1.1). We need only to verify Axiom (4.1.2). The arguments above in this proof, properly re-phrased, verify Axiom (4.1.2) for the X-surfaces $C_{-+}(\alpha;\beta)$, $C_{--}(\alpha;1)$, and $D_{--+}(\alpha,\beta;1,1)$. Since $C_{++} = -C_{--}$ and (4.1.2) for W is equivalent to (4.1.2) for $-W$, this axiom holds also for $W = C_{++}(\alpha;1)$. For $W = B_+$, we have $-W = B_-$ and (4.1.2) is equivalent to

$$\langle 1_L,b\rangle_1 = \langle \tau(B_+)(1_K),b\rangle_1 = \langle 1_K,\tau(B_-)(b)\rangle_\emptyset = \overline{\tau(B_-)(b)} \tag{5.1.c}$$

for any $b \in L_1$. By (4.1.e), $\tau(B_-)(b) = \eta(b,1_L)$. Therefore (5.1.c) is equivalent to the equality $\langle 1_L,b\rangle_1 = \overline{\eta(b,1_L)}$. Applying the conjugation in K, we obtain an equivalent formula $\langle b,1_L\rangle_1 = \eta(b,1_L)$. Since $\langle b,1_L\rangle_1 = \eta(b,\overline{1_L})$, the latter formula follows from the equality $\overline{1_L} = 1_L$.

Every X-surface W can be obtained by gluing from some X-surfaces of types

$$B_+,\ \ C_{-+}(\alpha;\beta),\ \ C_{--}(\alpha;1),\ \ C_{++}(\alpha;1),\ \ D_{--+}(\alpha,\beta;1,1).$$

Therefore Axiom (4.1.2) holds for W. Thus, the functor $HQ_2(X) \to \mathcal{HQ}(G)$ is surjective on objects. That this functor is an equivalence of categories directly follows from Theorem 3.1. □

5.2 Corollary. *The underlying G-algebra of a unitary two-dimensional HQFT with target $X = K(G,1)$ is a unitary crossed Frobenius G-algebra. This establishes an equivalence between the category $UQ_2(X)$ of two-dimensional unitary X-HQFTs and the category $\mathfrak{UQ}(G)$ of unitary crossed Frobenius G-algebras.*

Chapter IV
Biangular algebras and lattice HQFTs

IV.1 Frobenius G-algebras re-examined

We show in this section that every Frobenius G-algebra admits a natural family of endomorphisms similar to the family $\{\varphi_\alpha\}_{\alpha\in G}$ in the definition of a crossed G-algebra. This will lead us to a notion of a biangular algebra formulated in the next section. We begin with a simple algebraic lemma.

1.1 Lemma. *Let P and Q be projective K-modules of finite type and let $\eta\colon P\otimes Q\to K$ be a non-degenerate bilinear form. There is a unique vector $\eta^-\in P\otimes Q$ such that for any expansion $\eta^-=\sum_i p_i\otimes q_i$ into a finite sum with $p_i\in P$, $q_i\in Q$ and any $p\in P$, $q\in Q$,*

$$p=\sum_i \eta(p,q_i)\,p_i \quad \textit{and} \quad q=\sum_i \eta(p_i,q)\,q_i. \tag{1.1.a}$$

For $\eta^-=\sum_i p_i\otimes q_i$ satisfying (1.1.a), the formula $k\mapsto k\sum_i q_i\otimes p_i$ defines a homomorphism $K\to Q\otimes P$ satisfying (together with η) the conditions of Lemma I.5.1. Lemma 1.1 complements Lemma I.5.1 by saying that η gives rise to a unique η^- as in Lemma I.5.1.

Proof. Define a homomorphism $\rho\colon Q\to P^*$ by $\rho(q)(p)=\eta(p,q)$ for any $p\in P$ and $q\in Q$. Since η is non-degenerate, ρ is an isomorphism. For any $r=\sum_i p_i\otimes q_i\in P\otimes Q$, we define a K-homomorphism $\nu_r\colon P^*\to Q$ carrying $x\in P^*$ to $\sum_i x(p_i)\,q_i\in Q$. Since P and Q are projective K-modules of finite type, the formula $r\mapsto \nu_r$ defines an isomorphism $P\otimes Q\cong \mathrm{Hom}_K(P^*,Q)$.

Given $r=\sum_i p_i\otimes q_i\in P\otimes Q$, the identity $p=\sum_i\eta(p,q_i)p_i$ for all $p\in P$ is equivalent to the equality $\rho\,\nu_r=\mathrm{id}_{P^*}$. Indeed, $\rho\,\nu_r$ carries $x\in P^*$ into the homomorphism $P\to K$ mapping $p\in P$ to $\eta\big(p,\sum_i x(p_i)q_i\big)=x\big(\sum_i\eta(p,q_i)p_i\big)$. The equality $\rho\,\nu_r(x)=x$ holds for all $x\in P^*$ if and only if $\sum_i\eta(p,q_i)p_i=p$ for all $p\in P$. Similarly, the identity $q=\sum_i\eta(p_i,q)q_i$ for all $q\in Q$ is equivalent to the equality $\nu_r\,\rho=\mathrm{id}_Q$. Thus, the only vector $r=\eta^-$ satisfying the conditions of the lemma is determined from the formula $\nu_r=\rho^{-1}\in\mathrm{Hom}_K(P^*,Q)$. □

1.2 Operators $\{\psi_\alpha\}_{\alpha\in G}$. Let $(B=\bigoplus_{\alpha\in G}B_\alpha,\eta)$ be a Frobenius G-algebra over the ring K. By Lemma 1.1, for $\alpha\in G$, the non-degenerate form $\eta\colon B_\alpha\otimes B_{\alpha^{-1}}\to K$ yields a vector $\eta_\alpha^-\in B_\alpha\otimes B_{\alpha^{-1}}$. We expand

$$\eta_\alpha^-=\sum_i p_i^\alpha\otimes q_i^\alpha, \tag{1.2.a}$$

where i runs over a finite set of indices, J_α, and $p_i^\alpha \in B_\alpha, q_i^\alpha \in B_{\alpha^{-1}}$. The vector η_α^- is characterized by the following property: for any $a \in B_\alpha$,

$$a = \sum_{i \in J_\alpha} \eta(a, q_i^\alpha)\, p_i^\alpha.$$

Next, we define a K-linear homomorphism $\psi_\alpha : B \to B$ by

$$\psi_\alpha(b) = \sum_{i \in J_\alpha} p_i^\alpha b q_i^\alpha$$

for any $b \in B$. Clearly, the right-hand side does not depend on the choice of the expansion (1.2.a) of η_α^-. If $b \in B_\beta$, then $\psi_\alpha(b) \in B_{\alpha\beta\alpha^{-1}}$.

Since η is symmetric, $\eta_{\alpha^{-1}}^-$ is obtained from η_α^- by permutation of the tensor factors, i.e.,

$$\sum_{j \in J_{\alpha^{-1}}} p_j^{\alpha^{-1}} \otimes q_j^{\alpha^{-1}} = \sum_{i \in J_\alpha} q_i^\alpha \otimes p_i^\alpha. \tag{1.2.b}$$

Therefore

$$\psi_{\alpha^{-1}}(b) = \sum_{j \in J_{\alpha^{-1}}} p_j^{\alpha^{-1}} b q_j^{\alpha^{-1}} = \sum_{i \in J_\alpha} q_i^\alpha b p_i^\alpha. \tag{1.2.c}$$

In general, the endomorphisms $\{\psi_\alpha\}_{\alpha \in G}$ of B are neither multiplicative nor bijective. The next lemma exhibits their main properties. Recall that for $c \in B$, the symbol μ_c denotes the homomorphism $B \to B, b \mapsto cb$.

1.3 Lemma. *For any $\alpha, \beta \in G$ and $a, b \in B$,*

$$\eta(\psi_\alpha(a), b) = \eta(a, \psi_{\alpha^{-1}}(b)), \tag{1.3.a}$$

$$\psi_\alpha(a\psi_\beta(b)) = \psi_\alpha(a)\, \psi_{\alpha\beta}(b), \tag{1.3.b}$$

$$\psi_\alpha\, \psi_\beta(b) = \psi_\alpha(1_B)\, \psi_{\alpha\beta}(b). \tag{1.3.c}$$

For any $\alpha, \beta \in G$ and $b \in B_\beta$,

$$\psi_{\alpha\beta}(b) = \psi_\alpha(b). \tag{1.3.d}$$

For any $\alpha, \beta \in G$, $a \in B$, and $b \in B_\beta$,

$$\psi_\alpha(a)\, b = b\, \psi_{\beta^{-1}\alpha}(a). \tag{1.3.e}$$

For any $\alpha, \beta \in G$ and $c \in B_{\alpha\beta\alpha^{-1}\beta^{-1}}$,

$$\mathrm{Tr}(\mu_c\, \psi_\beta : B_\alpha \to B_\alpha) = \mathrm{Tr}(\psi_{\alpha^{-1}} \mu_c : B_\beta \to B_\beta). \tag{1.3.f}$$

Proof. We prepare a few preliminary formulas. First of all, the expression $\eta(ab,c)$ with $a,b,c \in B$ is invariant under cyclic permutations of a,b,c. Indeed, $\eta(ab,c) = \eta(a,bc) = \eta(bc,a)$. Secondly, for any $\alpha \in G$, $a \in B_{\alpha^{-1}}$, $a' \in B_\alpha$,

$$\begin{aligned}\sum_{i\in J_\alpha} \eta(a,p_i^\alpha)\,\eta(q_i^\alpha,a') &= \sum_{i\in J_\alpha} \eta(a,p_i^\alpha)\,\eta(a',q_i^\alpha)\\ &= \eta\big(a,\sum_{i\in J_\alpha} \eta(a',q_i^\alpha)\,p_i^\alpha\big) = \eta(a,a').\end{aligned} \tag{1.3.g}$$

Thirdly, for any $b \in B_{\beta^{-1}}$,

$$\sum_{i\in J_\alpha} p_i^\alpha \otimes q_i^\alpha b = \sum_{j\in J_{\beta\alpha}} bp_j^{\beta\alpha} \otimes q_j^{\beta\alpha}. \tag{1.3.h}$$

Note that both sides belong to $B_\alpha \otimes B_{\alpha^{-1}\beta^{-1}}$. It suffices to prove that they determine equal functionals on the dual K-module $B_{\alpha^{-1}} \otimes B_{\beta\alpha}$. Pick $x \in B_{\alpha^{-1}}$ and $y \in B_{\beta\alpha}$. By (1.3.g), the left-hand side of (1.3.h) evaluated on $x \otimes y$ gives

$$\sum_i \eta(p_i^\alpha,x)\,\eta(q_i^\alpha b,y) = \sum_i \eta(x,p_i^\alpha)\,\eta(q_i^\alpha,by) = \eta(x,by).$$

Similarly, the right-hand side of (1.3.h) evaluated on $x \otimes y$ gives

$$\sum_j \eta(bp_j^{\beta\alpha},x)\,\eta(q_j^{\beta\alpha},y) = \sum_j \eta(xb,p_j^{\beta\alpha})\,\eta(q_j^{\beta\alpha},y) = \eta(xb,y) = \eta(x,by).$$

This proves (1.3.h). Formula (1.3.h) implies that for any $a \in B$ and $b \in B_{\beta^{-1}}$,

$$\sum_{i\in J_\alpha} p_i^\alpha a q_i^\alpha b = \sum_{j\in J_{\beta\alpha}} bp_j^{\beta\alpha} a q_j^{\beta\alpha}. \tag{1.3.i}$$

Exchanging α and β, and replacing a,b with b,c, respectively, we obtain that for any $b \in B$, $c \in B_{\alpha^{-1}}$,

$$\sum_{j\in J_\beta} p_j^\beta b q_j^\beta c = \sum_{k\in J_{\alpha\beta}} c\, p_k^{\alpha\beta} b q_k^{\alpha\beta}. \tag{1.3.j}$$

We can now prove formulas (1.3.a)–(1.3.f). By cyclic symmetry,

$$\begin{aligned}\eta(\psi_\alpha(a),b) &= \eta\big(\sum_{i\in J_\alpha} p_i^\alpha a q_i^\alpha, b\big)\\ &= \sum_{i\in J_\alpha} \eta(aq_i^\alpha b, p_i^\alpha) = \eta\big(a, \sum_{i\in J_\alpha} q_i^\alpha b p_i^\alpha\big) = \eta(a,\psi_{\alpha^{-1}}(b)).\end{aligned}$$

This proves (1.3.a).

Applying (1.3.j) to $c = q_i^\alpha \in B_{\alpha^{-1}}$, we obtain

$$\psi_\alpha(a\psi_\beta(b)) = \sum_{\substack{i\in J_\alpha\\ j\in J_\beta}} p_i^\alpha a p_j^\beta b q_j^\beta q_i^\alpha = \sum_{\substack{i\in J_\alpha,\\ k\in J_{\alpha\beta}}} p_i^\alpha a q_i^\alpha p_k^{\alpha\beta} b q_k^{\alpha\beta} = \psi_\alpha(a)\,\psi_{\alpha\beta}(b).$$

This proves (1.3.b). Substituting $a = 1_B$ in (1.3.b), we obtain (1.3.c).

Formula (1.3.h) implies that for any $b \in B_{\beta^{-1}}$,

$$\psi_{\alpha^{-1}}(b) = \sum_{i \in J_\alpha} q_i^\alpha b p_i^\alpha = \sum_{j \in J_{\beta\alpha}} q_j^{\beta\alpha} b p_j^{\beta\alpha} = \psi_{(\beta\alpha)^{-1}}(b) = \psi_{\alpha^{-1}\beta^{-1}}(b).$$

This is equivalent to (1.3.d).

Replacing β by β^{-1} in (1.3.i) we obtain a formula equivalent to (1.3.e).

We now check (1.3.f). By Lemma I.5.1 and formula (1.2.c),

$$\begin{aligned}\mathrm{Tr}(\mu_c\, \psi_\beta \colon B_\alpha \to B_\alpha) &= \sum_{i \in J_\alpha} \eta(c\psi_\beta(p_i^\alpha), q_i^\alpha) \\ &= \sum_{\substack{i \in J_\alpha \\ j \in J_\beta}} \eta(c p_j^\beta\, p_i^\alpha q_j^\beta, q_i^\alpha) \\ &= \sum_{\substack{i \in J_\alpha \\ j \in J_\beta}} \eta(q_i^\alpha c p_j^\beta\, p_i^\alpha, q_j^\beta) \\ &= \sum_{j \in J_\beta} \eta(\psi_{\alpha^{-1}}(c p_j^\beta), q_j^\beta) = \mathrm{Tr}(\psi_{\alpha^{-1}}\mu_c \colon B_\beta \to B_\beta). \quad \square\end{aligned}$$

1.4 Lemma. *Let $\alpha \in G$. Then we have $\psi_\alpha(1_B) = 1_B$ if and only if $\eta(\ell, 1_B) = \mathrm{Tr}(\mu_\ell|_{B_\alpha} \colon B_\alpha \to B_\alpha)$ for all $\ell \in B_1$.*

Proof. Clearly, $\psi_\alpha(1_B) = \sum_{i \in J_\alpha} p_i^\alpha q_i^\alpha$. By Lemma I.5.1, for $\ell \in B_1$,

$$\eta(\ell, \psi_\alpha(1_B)) = \eta(\ell, \sum_{i \in J_\alpha} p_i^\alpha q_i^\alpha) = \sum_{i \in J_\alpha} \eta(\ell\, p_i^\alpha, q_i^\alpha) = \mathrm{Tr}(\mu_\ell|_{B_\alpha} \colon B_\alpha \to B_\alpha).$$

Therefore $\psi_\alpha(1_B) = 1_B$ if and only if $\eta(\ell, 1_B) = \mathrm{Tr}(\mu_\ell|_{B_\alpha})$ for all $\ell \in B_1$. $\square$

Lemma 1.4 implies that $\psi_\alpha(1_B) = 1_B$ for all $\alpha \in G$ if and only if B satisfies the following two conditions:

(1.4.1) for any $\ell \in B_1$ and any $\alpha \in G$,

$$\mathrm{Tr}(\mu_\ell|_{B_\alpha} \colon B_\alpha \to B_\alpha) = \mathrm{Tr}(\mu_\ell|_{B_1} \colon B_1 \to B_1),$$

(1.4.2) for any $\ell \in B_1$, we have $\eta(\ell, 1) = \mathrm{Tr}(\mu_\ell|_{B_1} \colon B_1 \to B_1)$.

Condition (1.4.1) means that considered as left B_1-modules, all B_α give rise to the same trace function on B_1. In particular, for $\alpha \in G$,

$$\begin{aligned}\mathrm{Dim}\, B_\alpha &= \mathrm{Tr}(\mathrm{id} \colon B_\alpha \to B_\alpha) \\ &= \mathrm{Tr}(\mu_1|_{B_\alpha} \colon B_\alpha \to B_\alpha) \\ &= \mathrm{Tr}(\mu_1|_{B_1} \colon B_1 \to B_1) \\ &= \mathrm{Tr}(\mathrm{id} \colon B_1 \to B_1) = \mathrm{Dim}\, B_1.\end{aligned}$$

Condition (1.4.1) is met if all B_α are isomorphic to B_1 as left B_1-modules.

Condition (1.4.2) implies that the inner product η on B is determined by multiplication in B. Namely, for $a \in B_\alpha$ and $b \in B_\beta$,

$$\eta(a,b) = \begin{cases} 0 & \text{if } \alpha\beta \neq 1, \\ \operatorname{Tr}(\mu_{ab}|_{B_1} : B_1 \to B_1) & \text{if } \alpha\beta = 1. \end{cases} \tag{1.4.a}$$

IV.2 Biangular G-algebras

2.1 Definition. A G-algebra $B = \bigoplus_{\alpha\in G} B_\alpha$ is *biangular* if all B_α are projective K-modules of finite type satisfying (1.4.1) and formula (1.4.a) defines an inner product η on B. We call η the *canonical inner product* on B. The condition on η amounts to saying that the restriction of η to $B_\alpha \otimes B_{\alpha^{-1}}$ is non-degenerate for all $\alpha \in G$; all other requirements on an inner product follow from (1.4.a) and the properties of the trace. Note that $\eta(1_B, 1_B) = \operatorname{Dim} B_1 = \operatorname{Dim} B_\alpha$ for all $\alpha \in G$.

A biangular G-algebra B with canonical inner product is a Frobenius G-algebra. By Lemma 1.4, the associated homomorphisms $\psi_\alpha : B \to B$ carries 1_B to itself for all $\alpha \in G$. Using the expansion (1.2.a) of η_α^-, we can rewrite this as

$$\sum_{i\in J_\alpha} p_i^\alpha q_i^\alpha = 1_B. \tag{2.1.a}$$

If a G-algebra $B = \bigoplus_{\alpha\in G} B_\alpha$ is biangular, then the algebra B_1 is non-degenerate in the following sense. A unital K-algebra C is *non-degenerate* if its underlying K-module is projective of finite type and the bilinear form $\eta : C \otimes C \to K$ defined by

$$\eta(a,b) = \operatorname{Tr}(\mu_{ab} : C \to C) \quad \text{for } a, b \in C$$

is non-degenerate. Examples of non-degenerate algebras are provided by the matrix rings $\operatorname{Mat}_n(K)$ such that n is invertible in K. If K is an algebraically closed field, then all finite-dimensional non-degenerate K-algebras split as finite direct sums of such matrix rings.

The next lemma associates with any biangular G-algebra B a subalgebra L of B called the *G-center* of B. Generally speaking, L is not commutative and differs from the usual center of the algebra B. On the other hand, L has a natural structure of a crossed Frobenius G-algebra. By Theorem III.3.1, L determines an isomorphism class of 2-dimensional HQFTs with target $K(G,1)$. An explicit construction of an HQFT in this class will be given in Section 3.

2.2 Lemma. *Let $B = \bigoplus_{\alpha\in G} B_\alpha$ be a biangular G-algebra with canonical inner product η and the associated homomorphisms $\{\psi_\alpha : B \to B\}_{\alpha\in G}$. Set $L = \bigoplus_{\alpha\in G} L_\alpha$ where $L_\alpha = \psi_1(B_\alpha) \subset B_\alpha$. Then*

(i) *L is a subalgebra of B and $\psi_\alpha(L) = L$ for all $\alpha \in G$;*

(ii) *the triple $(L, \eta|_L, \{\psi_\alpha|_L\}_{\alpha\in G})$ is a crossed Frobenius G-algebra.*

Proof. Since $\psi_1(1_B) = 1_B$, formula (1.3.c) with $\alpha = \beta = 1$ implies that $\psi_1\psi_1 = \psi_1 \colon B \to B$. By (1.3.a), the homomorphism ψ_1 is self-adjoint with respect to η. This implies that B splits as an orthogonal sum of $L = \operatorname{Im}\psi_1$ and $\operatorname{Ker}\psi_1$. (Any $a \in B$ splits as $a = \psi_1(a) + (a - \psi_1(a))$.) Therefore all L_α are projective K-modules of finite type. Clearly, $\psi_1|_L = \mathrm{id}_L \colon L \to L$.

Formula (1.3.b) with $\alpha = \beta = 1$ shows that L is a subalgebra of B. It is clear that $1_B = \psi_1(1_B) \in L$ is a unit of L and the restriction of η to L is an inner product on L. Hence L is a Frobenius G-algebra.

Formula (1.3.c) with $\alpha = 1$ shows that $\psi_\beta(B) \subset L$ for all $\beta \in G$. Set $\varphi_\beta = \psi_\beta|_L \colon L \to L$. Formula (1.3.c) with $\alpha = \beta^{-1}$ implies that φ_β and $\varphi_{\beta^{-1}}$ are mutually inverse endomorphisms of L. These endomorphisms are algebra homomorphisms, as follows from formula (1.3.b) with $\beta = 1$. Using (1.3.c), we obtain that $\varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta$ for all $\alpha, \beta \in G$. Formula (1.3.a) shows that φ_α preserves the inner product $\eta|_L$ on L. Formula (1.3.d) with $\alpha = 1$ implies that φ_β acts as the identity on L_β. Formula (1.3.e) with $\alpha = \beta$ implies that $\varphi_\beta(a)b = ba$ for all $a \in L$ and $b \in L_\beta$. Finally, for any $\alpha, \beta \in G$ and $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$,

$$\begin{aligned}\operatorname{Tr}(\mu_c\,\varphi_\beta \colon L_\alpha \to L_\alpha) &= \operatorname{Tr}(\mu_c\,\psi_\beta \colon L_\alpha \to L_\alpha)\\ &= \operatorname{Tr}(\mu_c\,\psi_\beta \colon B_\alpha \to B_\alpha)\\ &= \operatorname{Tr}(\psi_{\alpha^{-1}}\mu_c \colon B_\beta \to B_\beta)\\ &= \operatorname{Tr}(\psi_{\alpha^{-1}}\mu_c \colon L_\beta \to L_\beta)\\ &= \operatorname{Tr}(\varphi_{\alpha^{-1}}\mu_c \colon L_\beta \to L_\beta),\end{aligned}$$

where the second and fourth equalities follow from the inclusion $\psi_\beta(B) \subset L$ and the third equality is (1.3.f). Thus L satisfies all requirements on a crossed Frobenius G-algebra. □

2.3 Examples and constructions. 1. The group ring $B = K[G]$ with canonical G-algebra structure is biangular. Here $\psi_1 = \mathrm{id}$ so that the G-center of B is B itself. The next three examples generalize this one.

2. Let $q \colon G' \to G$ be a group epimorphism with finite kernel Γ whose order is invertible in K. Let $B = K[G']$ be the group algebra of G' viewed as a G-algebra as in Example II.1.2.1. It is easy to check from the definitions that the G-algebra B is biangular. The canonical inner product η on B is computed on the generators $a, b \in G' \subset B$ by $\eta(a, b) = |\Gamma|$ if $a = b^{-1}$ and $\eta(a, b) = 0$ otherwise. For $\alpha \in G$, the homomorphism $\psi_\alpha \colon B \to B$ is computed on any $b \in B$ by

$$\psi_\alpha(b) = |\Gamma|^{-1} \sum_{a \in q^{-1}(\alpha)} aba^{-1}.$$

In particular, the projection $\psi_1\colon B \to B$ on the G-center of B acts by

$$\psi_1(b) = |\Gamma|^{-1} \sum_{a\in\Gamma} aba^{-1}.$$

If Γ is central in G', then $\psi_1 = \mathrm{id}$ and B coincides with its G-center. This explains why B is crossed in this case (cf. Example II.3.3.1). For $q = \mathrm{id}_G\colon G \to G$, we recover Example 1.

3. Consider the G-algebra $B = \bigoplus_{\alpha\in G} K l_\alpha$ derived from a normalized 2-cocycle $\{\theta_{\alpha,\beta} \in K^*\}_{\alpha,\beta\in G}$ as in Example II.1.2.1. The G-algebra B is easily seen to be biangular. The canonical inner product η on B is computed by $\eta(l_\alpha, l_\beta) = \theta_{\alpha,\beta}$ if $\alpha = \beta^{-1}$ and $\eta(\alpha,\beta) = 0$ otherwise, cf. Section II.2.3.2. For $\alpha \in G$, the homomorphism $\psi_\alpha\colon B \to B$ acts on any l_β by

$$\psi_\alpha(l_\beta) = \theta^{-1}_{\alpha,\alpha^{-1}}\, l_\alpha\, l_\beta\, l_{\alpha^{-1}} = \theta^{-1}_{\alpha,\alpha^{-1}}\, \theta_{\alpha,\beta}\, \theta_{\alpha\beta,\alpha^{-1}}\, l_{\alpha\beta\alpha^{-1}}.$$

Hence $\psi_1 = \mathrm{id}$ and the G-center of B is B. This explains the existence of a crossed structure on B in Example II.3.3.2.

4. The previous examples have the following common generalization. Let $q\colon G' \to G$ be a group epimorphism with finite kernel Γ whose order is invertible in K. Fix a normalized 2-cocycle $\{\theta_{a,b} \in K^*\}_{a,b\in G'}$ on G'. Let $B = \bigoplus_{a\in G'} K l_a$ be the free K-module with basis $\{l_a\}_{a\in G'}$ and with multiplication $l_a\, l_b = \theta_{a,b}\, l_{ab}$ for $a, b \in G'$. Given $\alpha \in G$, set $B_\alpha = \bigoplus_{a\in q^{-1}(\alpha)} K l_a \subset B$. Clearly, $B = \bigoplus_{\alpha\in G} B_\alpha$ is a G-algebra. It is biangular. In particular, Condition (1.4.1) follows from the fact that for any $\alpha \in G$ and $b \in \Gamma$,

$$\mathrm{Tr}(\mu_{\ell_b}|_{B_\alpha}\colon B_\alpha \to B_\alpha) = \begin{cases} 0 & \text{if } b \neq 1, \\ |\Gamma| & \text{if } b = 1. \end{cases}$$

The canonical inner product η on B is computed on the generators l_a, l_b with $a, b \in G'$ by $\eta(l_a, l_b) = \theta_{a,b}\, |\Gamma|$ if $a = b^{-1}$ and $\eta(l_a, l_b) = 0$ otherwise. For $\alpha \in G$, the vector $\eta^-_\alpha \in B_\alpha \otimes B_{\alpha^{-1}}$ defined in Lemma 1.1 is computed as follows:

$$\eta^-_\alpha = |\Gamma|^{-1} \sum_{a\in q^{-1}(\alpha)} \theta^{-1}_{a,a^{-1}}\, l_a \otimes l_{a^{-1}}. \tag{2.3.a}$$

The homomorphism $\psi_\alpha\colon B \to B$ acts on l_b with $b \in G'$ by

$$\psi_\alpha(l_b) = |\Gamma|^{-1} \sum_{a\in q^{-1}(\alpha)} \theta^{-1}_{a,a^{-1}}\, l_a\, l_b\, l_{a^{-1}} = |\Gamma|^{-1} \sum_{a\in q^{-1}(\alpha)} \theta^{-1}_{a,a^{-1}}\, \theta_{a,b}\, \theta_{ab,a^{-1}}\, l_{aba^{-1}}.$$

Note that $l_a \in B$ is invertible in B and $l_a^{-1} = \theta^{-1}_{a,a^{-1}}\, l_{a^{-1}}$. This implies that for any $l \in B$,

$$\psi_\alpha(l) = |\Gamma|^{-1} \sum_{a\in q^{-1}(\alpha)} l_a\, l\, l_a^{-1}. \tag{2.3.b}$$

5. Let A be a non-degenerate algebra over K. Suppose that G acts on A by algebra automorphisms. Consider the direct sum $\mathcal{A} = \bigoplus_{\alpha \in G} A_\alpha$, where each A_α is a copy of A labeled by α. We provide $\mathcal{A}$ with associative multiplication $*$ by $a * b = a\alpha(b)$, where $a \in A_\alpha = A$, $b \in A_\beta = A$ with $\alpha, \beta \in G$ and $a\alpha(b) \in A_{\alpha\beta} = A$ is the product of a and $\alpha(b)$ in A. It is straightforward to check that $\mathcal{A}$ is a biangular G-algebra.

6. Further examples of biangular G-algebras may be constructed using the standard operations on G-algebras. In particular, direct sums and tensor products of biangular G-algebras are biangular G-algebras. The pull-back of a biangular G-algebra along a group homomorphism $G' \to G$ is a biangular G'-algebra.

2.4 Lemma. *Let $B = \bigoplus_{\alpha \in G} B_\alpha$ be a biangular G-algebra over an algebraically closed field K. Then the G-center of B is semisimple in the sense of Section* II.5.1.

Proof. Since the K-algebra B_1 is non-degenerate, B_1 is a direct sum of matrix rings over K. This allows us to compute explicitly the canonical inner product η on B and the associated projector $\psi_1 \colon B \to B$. If B_1 is a matrix ring, $B_1 = \mathrm{Mat}_n(K)$ with $n \geq 1$, then $\eta(a,b) = n\,\mathrm{Tr}(ab)$ for all $a, b \in B_1 = \mathrm{Mat}_n(K)$. Since B_1 is non-degenerate, n must be invertible in K. The vector $\eta_1^- \in B_1 \otimes B_1$ associated with $\eta_1 = \eta|_{B_1}$ via Lemma 1.1 is equal to $n^{-1} \sum_{i,j=1}^{n} e_{i,j} \otimes e_{j,i}$, where $e_{i,j}$ is the elementary $(n \times n)$-matrix whose (i,j)-term is 1 and all other terms are zero. The homomorphism $\psi_1 \colon B_1 \to B_1$ carries any $a \in B_1$ to $n^{-1} \sum_{i,j=1}^{n} e_{i,j} a e_{j,i} = n^{-1}\,\mathrm{Tr}(a) E_n$, where E_n is the unit $(n \times n)$-matrix. Thus, ψ_1 is a projection of $\mathrm{Mat}_n(K)$ onto its 1-dimensional center. If B_1 is a direct sum of m matrix rings then ψ_1 is a projection of B_1 onto its m-dimensional center K^m. Hence $\psi_1(B_1) = K^m$. Thus the crossed Frobenius G-algebra $\psi_1(B)$ is semisimple. □

Note that if the characteristic of the field K in Lemma 2.4 is zero, then the isomorphism type of the G-center $L = \psi_1(B)$ of B is entirely determined by the basic triple (I, ∇, F) (Theorem II.6.3). Here $I = \mathrm{Bas}(L) \subset B_1$ is the G-set of the identity elements of the direct summands $\mathrm{Mat}_n(K)$ of B_1 (the algebra B_1 is a direct sum of matrix rings over K, cf. the proof of Lemma 2.4). The action of G on I is given by $\alpha e = \psi_\alpha(e)$ for $\alpha \in G$ and $e \in I$. If $e \in I$ is the identity element of a direct summand $\mathrm{Mat}_n(K)$ of B_1, then

$$F(e) = \eta(e,e) = \mathrm{Tr}(\mu_e \colon B_1 \to B_1) = n^2.$$

The equivariant cohomology class ∇ associates to every $e \in I$ the cohomology class of the following K^*-valued 2-cocycle $\{\nabla_{\alpha,\beta}\}_{\alpha,\beta}$ on the group $G_e = \{\alpha \in G \mid \psi_\alpha(e) = e\}$ (cf. Section II.5). For each $\alpha \in G_e$, pick a non-zero vector $s_\alpha \in eL_\alpha \cong K$. Then $\nabla_{\alpha,\beta} \in K^*$ is defined from the equality $s_\alpha s_\beta = \nabla_{\alpha,\beta}\, s_{\alpha\beta}$.

IV.3 Lattice HQFTs

Let X be an aspherical connected CW-space with base point x and fundamental group $G = \pi_1(X, x)$. We describe a state sum model on X-surfaces that derives from any biangular G-algebra a two-dimensional X-HQFT. This generalizes the state sum model of C. Bachas, P. Petropoulos [BP] and M. Fukuma, S. Hosono, H. Kawai [FHK] for a 2-dimensional TQFT ($G = 1$).

3.1 Maps to X as combinatorial systems. It is well known that the homotopy classes of maps from CW-complexes to $X = K(G, 1)$ admit a combinatorial description in terms of elements of G assigned to the 1-cells. We recall this description adapting it to our setting.

Let T be a CW-complex with underlying topological space $|T|$. By *vertices*, *edges*, and *faces* of T, we mean 0-cells, 1-cells, and 2-cells of T, respectively. Denote the set of vertices of T by $\mathrm{Vert}(T)$ and the set of *oriented* edges of T by $\mathrm{Edg}(T)$. Each oriented edge $e \in \mathrm{Edg}(T)$ leads from an *initial vertex* $i_e \in \mathrm{Vert}(T)$ to a *terminal vertex* $t_e \in \mathrm{Vert}(T)$ (they may coincide). The orientation reversal defines a free involution $e \mapsto e^{-1}$ on $\mathrm{Edg}(T)$.

A face Δ of T is obtained by adjoining a 2-disk to the 1-skeleton $T^{(1)}$ of T along a (continuous) map $f_\Delta \colon S^1 \to T^{(1)}$. In general f_Δ may be rather wild; for example, it may contract a subarc of S^1 to a point. We shall impose the following property of regularity. We say that T is *regular* if for any its face Δ, the set $f_\Delta^{-1}(\mathrm{Vert}(T)) \subset S^1$ is finite and non-empty and the restrictions of f_Δ to the components of $S^1 - f_\Delta^{-1}(\mathrm{Vert}(T))$ are injective. Then the points of $f_\Delta^{-1}(\mathrm{Vert}(T))$ split S^1 into arcs called the *sides* of Δ. The image $f_\Delta(e)$ of a side e of Δ is an edge of T called the *underlying edge* of e. By abuse of notation, we shall often denote the underlying edge of e by the same letter e. An orientation of S^1 induces an orientation and a cyclic order $e_1, e_2, \ldots, e_n$ on the sides of Δ so that the terminal endpoint of e_r is the initial endpoint of e_{r+1} for all $r \pmod n$, where $n \geq 1$ is the number of sides of Δ. The corresponding cyclically ordered oriented edges of T form the boundary of Δ. The opposite orientation of S^1 gives the cyclic sequence $e_n^{-1}, \ldots, e_2^{-1}, e_1^{-1}$. Examples of regular CW-complexes are provided by triangulated spaces. An example of a non-regular CW-complex is provided by the 2-sphere S^2 with CW-decomposition having one vertex, one face, and no edges.

Let T be a regular CW-complex with a fixed CW-subcomplex $T_\bullet$ (possibly empty). A *G-system* on T is a map $\mathrm{Edg}(T) \to G$, $e \mapsto g_e$ such that

(i) $g_{e^{-1}} = (g_e)^{-1}$ for any $e \in \mathrm{Edg}(T)$;

(ii) if ordered oriented edges $e_1, e_2, \ldots, e_n$ of T with $n \geq 1$ form the boundary of a face of T, then $g_{e_1} g_{e_2} \cdots g_{e_n} = 1$;

(iii) for any $e \in \mathrm{Edg}(T_\bullet) \subset \mathrm{Edg}(T)$, we have $g_e = 1$.

Note that Condition (ii) is preserved under cyclic permutations of $e_1, e_2, \ldots, e_n$ and under the inversion of orientation and cyclic order.

Two G-systems g, g' on T are *homotopic* if there is a map $\gamma\colon \operatorname{Vert}(T) \to G$ such that $\gamma(\operatorname{Vert}(T_\bullet)) = 1$ and $g'_e = \gamma(i_e)\, g_e\, (\gamma(t_e))^{-1}$ for all $e \in \operatorname{Edg}(T)$. If γ takes value 1 on all vertices of T except a vertex v, then we say that g' is obtained from g by a *homotopy move* at v. It is clear that two G-systems on T are homotopic if and only if they can be related by a finite sequence of homotopy moves at vertices of T not belonging to $T_\bullet$. Note that homotopy of G-systems on T is an equivalence relation.

Every homotopy class F of maps $(|T|, |T_\bullet|) \to (X, x)$ determines a homotopy class of G-systems g_F on T as follows: choose a map $f : (|T|, |T_\bullet|) \to (X, x)$ in the class F such that $f(\operatorname{Vert}(T)) = \{x\}$ and assign to each $e \in \operatorname{Edg}(T)$ the element of $G = \pi_1(X, x)$ represented by the loop $f|_e$. Conversely, Conditions (i)–(iii) and the asphericity of X imply that for any G-system g on T, there is a map $|g|\colon (|T|, |T_\bullet| \cup \operatorname{Vert}(T)) \to (X, x)$ that carries each $e \in \operatorname{Edg}(T)$ to a loop in X representing $g_e \in G$. The formulas $g \mapsto |g|$ and $F \mapsto g_F$ establish mutually inverse (and therefore bijective) correspondences between the homotopy classes of G-systems on T and the homotopy classes of maps $(|T|, |T_\bullet|) \to (X, x)$.

3.2 State sums on closed surfaces. Fix a biangular G-algebra $B = \bigoplus_{g\in G} B_g$ over K with canonical inner product η. Let W be a closed X-surface, i.e., a closed oriented surface endowed with a homotopy class of maps $W \to X$. We define a state sum invariant $\tau_B(W) \in K$ as follows.

Pick a regular CW-decomposition T of W. By a *flag* of T, we mean a pair (a face Δ of T, a side e of Δ). The flag (Δ, e) induces an orientation on e such that Δ lies on the right of e. This means that the pair (a vector looking from a point of e into Δ, the oriented edge of T underlying e) is positive with respect to the given orientation of W.

Let g be a G-system on T representing the given homotopy class of maps $|T| = W \to X$. With each flag (Δ, e) of T we associate the K-module $B(\Delta, e, g) = B_{g_e}$, where e is oriented so that Δ lies on its right.

Every edge e of T appears in two flags, $(\Delta_1, e), (\Delta_2, e)$, and inherits from them opposite orientations. Since the corresponding values of g are mutually inverse, Section 1.2 yields a vector $\eta^-_{g_e} \in \otimes_{i=1,2} B(\Delta_i, e, g)$. Formula (1.2.b) implies that this vector does not depend on the order in the 2-element set $\{\Delta_1, \Delta_2\}$. Set

$$\eta^-_g = \otimes_e\, \eta^-_{g_e} \in \otimes_{(\Delta,e)} B(\Delta, e, g),$$

where on the left-hand side e runs over all unoriented edges of T and on the right-hand side (Δ, e) runs over all flags of T.

Let Δ be a face of T with $n \geq 1$ sides. We orient and cyclically order the sides $e_1, e_2, \ldots, e_n$ of Δ so that Δ lies on their right; see Figure IV.1, where the plane of the picture is oriented clockwise. Then $g_{e_1} g_{e_2} \ldots g_{e_n} = 1$. The n-linear form

$$B(\Delta, e_1, g) \otimes B(\Delta, e_2, g) \otimes \cdots \otimes B(\Delta, e_n, g) \to K, \tag{3.2.a}$$

defined by

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto \eta(a_1 a_2 \cdots a_n, 1_B),$$

where $a_r \in B(\Delta, e_r, g)$ for $r = 1, 2, \ldots, n$, is invariant under cyclic permutations and therefore is well defined. The tensor product of these forms over all faces Δ of T is a homomorphism

$$D_g : \otimes_{(\Delta,e)} B(\Delta, e, g) \to K,$$

where (Δ, e) runs over all flags of T.

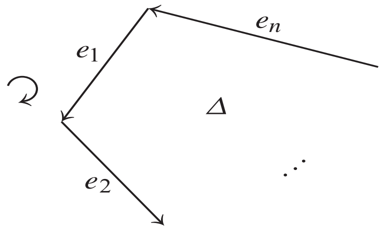


Figure IV.1. Oriented ordered edges of a face Δ.

3.2.1 Lemma. *The element $\langle g \rangle = D_g(\eta_g^-)$ of K does not depend on the choice of g in its homotopy class and does not depend on the choice of T.*

This lemma will be proved in Section 4. It implies that

$$\tau_B(W) = \langle g \rangle = D_g(\eta_g^-) \in K$$

is a well-defined invariant of the X-surface W. This invariant is multiplicative with respect to disjoint unions of surfaces. By definition, if $W = \emptyset$, then $\tau(W) = 1 \in K$.

The biangularity condition on B arises from the following argument. Consider a regular CW-decomposition T of W and an embedded arc $e_0 \subset W$ connecting a vertex v of T to a point w inside a face Δ and meeting the 1-skeleton $T^{(1)}$ only at v. Adding e_0 to $T^{(1)}$ we obtain a regular CW-decomposition T' of W with one additional vertex w, one additional edge e_0, and the face $\Delta' = \Delta - e_0$ replacing Δ. We say that T' is obtained from T by a stick addition. Given a G-system g on T and an arbitrary $\alpha \in G$, we can extend g uniquely to a G-system g^α on T' assigning α to the edge e_0 oriented from v to w. The form (3.2.a) associated with Δ' has two additional consecutive factors B_α and $B_{\alpha^{-1}}$. The edge e_0 contributes the vector $\eta_\alpha^- = \sum_i p_i^\alpha \otimes q_i^\alpha \in B_\alpha \otimes B_{\alpha^{-1}}$. To ensure that $\langle g \rangle = \langle g^\alpha \rangle$, it is enough to require that $\eta\big(\sum_i p_i^\alpha q_i^\alpha a, 1\big) = \eta(a, 1)$ for all $a \in B_1$. This is equivalent to the biangularity condition $\sum_i p_i^\alpha q_i^\alpha = 1$.

3.3 A lattice HQFT. We derive here from a biangular G-algebra B a two-dimensional X-HQFT (A_B, τ_B) extending the invariant of closed X-surfaces defined in the previous subsection. The construction proceeds in three steps. At the first step we assign K-modules to so-called trivialized X-curves. At the second step we extend this assignment to a preliminary HQFT $(A_B^\circ, \tau_B^\circ)$ defined for trivialized X-curves and X-surfaces with trivialized boundary. At the third step we get rid of the trivializations.

Step 1. Recall that a manifold M is pointed if every component of M is endowed with a base point. Recall that an X-curve is a pair (a pointed closed oriented 1-dimensional manifold M, a homotopy class of maps $M \to X$ carrying the base points of M to x). An X-curve M is *trivialized* if we are given

(i) a CW-decomposition T of M such that all base points of M belong to $\mathrm{Vert}(T)$. The set of base points of M is denoted $T_\bullet$;

(ii) a G-system g on T inducing the given homotopy class of maps $(|T|, |T_\bullet|) \to (X, x)$.

For a trivialized X-curve $M = (M, T, g)$, set $A^\circ_M = \otimes_e B_{g_e}$, where e runs over the edges of T and the orientation of e is induced by that of M. It is clear that A°_M is a projective K-module of finite type. For example, if $M = S^1$ and T has one vertex and one edge e, then $A^\circ_M = B_{g_e}$, where the orientation of e is induced by that of M.

For any trivialized X-curves M, N, we have $A^\circ_{M \amalg N} = A^\circ_M \otimes A^\circ_N$. By definition, $M = \emptyset$ is trivialized and $A^\circ_\emptyset = K$.

Step 2. Consider an X-surface W with bases M_0 and M_1 (cf. Section III.1.1). Thus, W is a compact oriented surface with pointed boundary $\partial W = M_0 \cup M_1$, where M_0, M_1 are disjoint closed oriented 1-dimensional manifolds with orientation of M_1 (resp. of M_0) induced by that of W (resp. of $-W$). Let $F : W \to X$ be the given homotopy class of maps carrying the base points of ∂W to x. The bases M_0 and M_1 of W become X-curves by restricting F. If both M_0 and M_1 are trivialized, then we define a K-homomorphism $\tau^\circ(W) : A^\circ_{M_0} \to A^\circ_{M_1}$ as follows. Pick a regular CW-decomposition T of W extending the given CW-decompositions of M_0 and M_1. Let $T_\bullet \subset \mathrm{Vert}(T)$ be the set of base points of ∂W. We call a G-system g on T *characteristic* if g extends the G-systems on the X-curves M_0 and M_1 provided by the trivializations and $|g| = F : (|T|, |T_\bullet|) \to (X, x)$. By Section 3.1, characteristic G-systems on T exist for all F.

Pick a characteristic G-system g on T. The notion of a flag (Δ, e) of T and the K-module $B(\Delta, e, g) = B_{g_e}$ defined in Section 3.2 in the case $\partial W = \emptyset$ generalizes to surfaces with boundary word for word. Each edge $e \subset \partial W$ extends to a unique flag (Δ_e, e). As in Section 3.2, we have the vector

$$\eta^-_g = \otimes_e \eta^-_{g_e} \in \otimes_{(\Delta, e \subset \mathrm{Int}\, W)} B(\Delta, e, g),$$

where e in $\otimes_e$ runs over all unoriented edges of T lying in $\mathrm{Int}\, W = W - \partial W$ and (Δ, e) in $\otimes_{(\Delta, e \subset \mathrm{Int}\, W)}$ runs over all flags of T such that $e \subset \mathrm{Int}\ W$. The same construction as in Section 3.2 defines a homomorphism $D_g : \otimes_{(\Delta, e)} B(\Delta, e, g) \to K$, where (Δ, e) runs over all flags of T. Contracting D_g with η^-_g, we obtain a homomorphism

$$\otimes_{e \subset M_0 \cup M_1} B(\Delta_e, e, g) \to K, \quad a \mapsto D_g(a \otimes \eta^-_g). \tag{3.3.a}$$

This homomorphism is denoted by $\langle g \rangle$.

We provide each edge e of T lying on M_r $(r = 0, 1)$ with the orientation induced by that of M_r. Since the latter is induced by the orientation of W for $r = 1$ and by the opposite orientation for $r = 0$, the face Δ_e of T adjacent to e lies on the left of e for $r = 1$ and on the right of e for $r = 0$. Therefore

$$B(\Delta_e, e, g) = \begin{cases} B_{g_e} & \text{if } e \subset M_0, \\ B_{g_e^{-1}} = (B_{g_e})^* & \text{if } e \subset M_1. \end{cases}$$

Here we identify $B_{g_e^{-1}}$ with the dual of B_{g_e} using the canonical inner product on B. In this way we can view $\langle g \rangle$ as a homomorphism

$$A^\circ_{M_0} = \bigotimes_{e \subset M_0} B_{g_e} \to \bigotimes_{e \subset M_1} B_{g_e} = A^\circ_{M_1}.$$

The next lemma will be proven in Section 4.

3.3.1 Lemma. *The homomorphism $\langle g \rangle$ does not depend on the choice of the characteristic system g and does not depend on the choice of T.*

Set $\tau^\circ(W) = \langle g \rangle \colon A^\circ_{M_0} \to A^\circ_{M_1}$. In particular, if $M_0 = M_1 = \emptyset$, then $\tau^\circ(W) \colon K \to K$ is multiplication by the invariant $\tau_B(W) \in K$ introduced in Section 3.2. We express this by writing $\tau^\circ(W) = \tau_B(W)$.

The next lemma describes the behavior of τ° under the gluing of X-surfaces.

3.3.2 Lemma. *Let M_0, M_1, N be trivialized X-curves. If an X-surface W with bases M_0 and M_1 is obtained from an X-surface W_0 with bases M_0, N and an X-surface W_1 with bases N, M_1 by gluing along N, then*

$$\tau^\circ(W) = \tau^\circ(W_1) \circ \tau^\circ(W_0) \colon A^\circ_{M_0} \to A^\circ_{M_1}.$$

Proof. For $r = 0, 1$, pick a regular CW-decomposition T_r of W_r extending the given CW-decompositions of the bases. Let $(T_r)_\bullet \subset T_r$ be the set of the base points of M_r and N. Gluing T_0 and T_1 along N we obtain a regular CW-decomposition T of W with $T_\bullet \subset T$ being the set of the base points of M_0 and M_1. Lemma 3.3.2 is a reformulation of the following claim:

Let g_r be a characteristic G-system on T_r for $r = 0, 1$. Let g be the unique G-system on T extending g_0 and g_1. Then g is a characteristic G-system on T and $\langle g \rangle \colon A^\circ_{M_0} \to A^\circ_{M_1}$ is the composition of $\langle g_0 \rangle \colon A^\circ_{M_0} \to A^\circ_N$ and $\langle g_1 \rangle \colon A^\circ_N \to A^\circ_{M_1}$.

This claim follows from the definitions. □

Step 3. The constructions above assign a K-module A°_M to any trivialized X-curve M and assign a homomorphism $\tau^\circ(W) \colon A^\circ_{M_0} \to A^\circ_{M_1}$ to any X-surface W with trivialized bases M_0 and M_1. This data looks like an HQFT and satisfies an appropriate version of Axioms (1.2.2)–(1.2.6) and (1.2.8). However, in general $\tau(M \times [0, 1]) \neq \mathrm{id}_{A^\circ_M}$. There is a standard procedure which transforms such a "pseudo-HQFT" (A°, τ°) into a genuine two-dimensional X-HQFT $(A, \tau) = (A_B, \tau_B)$ and gets rid of the trivializations at the same time. This procedure is described in detail in a similar setting in [Tu2], Section VII.3. The idea is that if t_0, t_1 are two trivializations of an X-curve M, then the cylinder cobordism $W = M \times [0,1]$ gives a homomorphism

$$p(t_0, t_1) = \tau^\circ(W) \colon A^\circ_{(M,t_0)} \to A^\circ_{(M,t_1)}.$$

By Lemma 3.3.2, $p(t_0, t_2) = p(t_1, t_2)\, p(t_0, t_1)$ for any trivializations t_0, t_1, t_2 of M. Taking $t_0 = t_1 = t_2$ we obtain that $p(t_0, t_0)$ is a projection onto a direct summand $A_{(M,t_0)}$ of $A^\circ_{(M,t_0)}$. Moreover, $p(t_0, t_1)$ maps $A_{(M,t_0)}$ isomorphically onto $A_{(M,t_1)}$. This allows us to identify the modules $\{A_{(M,t)}\}_t$ (where t runs over all trivializations of M) along these isomorphisms and to obtain a K-module A_M independent of t. To define the action of an X-homeomorphism $f: M \to M'$, we pick a trivialization t of M and consider the induced trivialization $t' = f(t)$ of M'. The homomorphism $f_\# \colon A_M \to A_{M'}$ is defined as the composition of the isomorphisms

$$A_M \cong A_{(M,t)} = A_{(M',t')} \cong A_{M'}.$$

Here the first and the third isomorphisms are the identification isomorphisms from the definition of A_M and $A_{M'}$ while the middle isomorphism is the identity map of the module $A_{(M,t)} = A_{(M',t')}$. It is easy to check that the homomorphism $f_\# \colon A_M \to A_{M'}$ does not depend on the choice of t.

Finally, we observe that for an X-surface W with trivialized bases M_0, M_1, the homomorphism $\tau^\circ(W) \colon A^\circ_{M_0} \to A^\circ_{M_1}$ maps $A_{M_0} \subset A^\circ_{M_0}$ into $A_{M_1} \subset A^\circ_{M_1}$. This yields a homomorphism $\tau(W) \colon A_{M_0} \to A_{M_1}$ independent of the trivializations of the bases.

3.3.3 Theorem. *The modules A_M and the homomorphisms $\tau(W)$ form a 2-dimensional X-HQFT. The underlying crossed Frobenius G-algebra of this HQFT is isomorphic to the G-center of B.*

Proof. The first claim directly follows from the definitions and Lemma 3.3.2. We verify the second claim. Let $(L' = \bigoplus_{\alpha \in G} L'_\alpha, \eta', \{\varphi'_\alpha\}_\alpha)$ be the crossed Frobenius G-algebra underlying the HQFT $(A, \tau) = (A_B, \tau_B)$. To compute L'_α for $\alpha \in G$, we represent α by a loop $f: S^1 \to X$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with clockwise orientation and base point $s = -i$. The X-curve $M = (S^1, f)$ has a canonical trivialization t formed by one vertex s, one edge e, and the G-system assigning α to the edge e oriented clockwise. By definition, $L'_\alpha = A_M$ is the image of the projector $P_M \colon A^\circ_{(M,t)} \to A^\circ_{(M,t)}$. Here $A^\circ_{(M,t)} = B_\alpha$ and $P_M = \tau(W, \bar{f} \colon W \to X)$, where $W = S^1 \times [0,1]$ is the cylinder cobordism between two copies of (M,t) and $\bar{f}$ is the composition of the projection $W \to S^1$ with $f: S^1 \to X$. Let T be the CW-decomposition of W formed by the vertices $(s,0)$, $(s,1)$, the edges $e_0 = e \times \{0\}$, $e_1 = e \times \{1\}$, $e_2 = \{s\} \times [0,1]$, and the face $(S^1 - \{s\}) \times (0,1)$. We orient e_0, e_1 clockwise and e_2 from $(s,0)$ to $(s,1)$. The map $\bar{f}$ is presented by the G-system g defined by $e_0 \mapsto \alpha$, $e_1 \mapsto \alpha$, and $e_2 \mapsto 1 \in G$. The X-surface $(W, \bar{f})$ is nothing but the annulus $C_{-+}(\alpha; 1)$ shown on the left part of Figure III.1 (where $\beta = 1$) together with the CW-decomposition T and the G-system g. It is clear from the definitions that $P_M = \tau^\circ(W, \bar{f}) = \langle g \rangle : B_\alpha \to B_\alpha$ carries any $a \in B_\alpha$ to $\sum_i p_i^1 a q_i^1$, where $\sum_i p_i^1 \otimes q_i^1 = \eta_1^- \in B_1 \otimes B_1$ is the vector associated with the canonical inner product η on B. Hence $P_M = \psi_1$ and $L'_\alpha = \operatorname{Im}(\psi_1 : B_\alpha \to B_\alpha) = L_\alpha$. This identifies L' as a K-module with the center L of B.

We now compute the inner product, the action of G, and the multiplication in $L' = L$ induced by (A, τ). Inverting the orientation of $S^1 \times \{1\} \subset \partial W$ (and keeping the map $\bar{f} : W \to X$ and the CW-decomposition T of W), we transform $(W, \bar{f})$ into an X-cobordism between $(M, t) \amalg (-M, t)$ and $\emptyset$. In the notation of Section III.1.2, this cobordism is $C_{--}(\alpha; 1)$. By definition,

$$\eta'_\alpha = \tau(W, \bar{f}) : L_\alpha \otimes L_{\alpha^{-1}} \to K$$

is the restriction of $\tau^\circ(W, \bar{f}) : B_\alpha \otimes B_{\alpha^{-1}} \to K$. For $a \in B_\alpha, b \in B_{\alpha^{-1}}$,

$$\tau^\circ(W, \bar{f})(a \otimes b) = \sum_i \eta(a p_i^1 b q_i^1, 1_B) = \eta(a \psi_1(b), 1_B) = \eta(a, \psi_1(b)).$$

Since $\psi_1|_L = \mathrm{id}$, we conclude that $\eta' = \eta|_{L \otimes L}$.

Consider again $W = S^1 \times [0, 1]$ as a cobordism between two copies of (M, t) with the same CW-decomposition T. For $\beta \in G$, the formulas $e_0 \mapsto \alpha$, $e_1 \mapsto \beta\alpha\beta^{-1}$, and $e_2 \mapsto \beta^{-1}$ define a G-system g^β on T representing a map $f^\beta : W \to X$. The X-surface (W, f^β) is the annulus $C_{-+}(\alpha; \beta^{-1})$ used to define φ'_β. By definition,

$$\varphi'_\beta|_{L_\alpha} = \tau(W, f^\beta) = \tau^\circ(W, f^\beta)|_{L_\alpha} = \langle g^\beta \rangle|_{L_\alpha} : L_\alpha \to L_{\beta\alpha\beta^{-1}}.$$

The homomorphism $\langle g^\beta \rangle : B_\alpha \to B_{\beta\alpha\beta^{-1}}$ carries any $a \in B_\alpha$ to $\sum_i p_i^\beta a q_i^\beta = \psi_\beta(a)$, where $\sum_i p_i^\beta \otimes q_i^\beta = \eta_\beta^- \in B_\beta \otimes B_{\beta^{-1}}$. Hence $\varphi'_\beta = \psi_\beta|_L$. Thus, the inner product and the action of G on L' and L coincide.

To compute multiplication $L'_\alpha \otimes L'_\beta \to L'_{\alpha\beta}$ in L', we use the CW-decomposition of the X-disk with two holes $D_{--+}(\alpha, \beta; 1, 1)$ shown in Figure III.2. This CW-decomposition has the vertices y, z, t, the edges ty, tz, three more edges parametrizing the boundary circles, and one face. A direct computation from the definitions shows that the product of $a \in L'_\alpha$ and $b \in L'_\beta$ in L' is equal to the product of $\psi_1(a)$ and $\psi_1(b)$ in B. However, $\psi_1(a) = a$ and $\psi_1(b) = b$ since $a, b \in L' = L$. Thus, multiplication in L' is the restriction of multiplication in B. We conclude that $L' = L$ as crossed Frobenius G-algebras. □

3.4 Remark. The construction $B \mapsto (A_B, \tau_B)$ transforms direct sums (resp. tensor products, pull-backs) of biangular G-algebras into direct sums (resp. tensor products, pull-backs) of HQFTs.

IV.4 Skeletons of surfaces

The aim of this section is to prove Lemmas 3.2.1 and 3.3.1. To this end, we first study skeletons of surfaces. Throughout this section, W is a compact oriented surface with pointed boundary.

4.1 Skeletons of W. By a *graph*, we mean a 1-dimensional CW-complex with finite number of vertices and edges. The *valency* of a vertex of a graph is the number of edges incident to this vertex counted with multiplicity.

A *skeleton* of W is a graph F embedded in W such that

(i) each component of $W - F$ is either an open 2-disk or a half-open 2-disk homeomorphic to $[0, 1) \times (0, 1)$ and meeting ∂W along $\{0\} \times (0, 1)$;

(ii) F has at least one vertex in the interior of each component of W and has no isolated vertices (i.e., vertices of valency 0);

(iii) each component N of ∂W meets F at a finite non-empty set consisting of 1-valent vertices of F and disjoint from the base point of N.

Condition (i) implies that the intersection of F with a connected component of W is connected. Condition (ii) excludes the case where W is a 2-disk and F is its diameter with no internal vertices and the case where $W = S^2$ and F is a point.

Note that a skeleton may have loops (i.e., edges with coinciding endpoints) and multiple edges (i.e., different edges with the same endpoints). It may also have 1-valent and 2-valent vertices lying in Int W.

A 1-valent vertex of a skeleton F of W lying on ∂W is called a *foot* of F. An open subset of F consisting of a foot and the adjacent open edge is called a *leg* of F.

For $\partial W = \emptyset$, an example of a skeleton of W is provided by the 1-skeleton of the CW-decomposition of W dual to a triangulation of W. This example will be generalized in Section 4.4.

4.2 Moves on skeletons. We define two local moves on a skeleton F of W transforming it into another skeleton of W with the same feet. The *contraction move* contracts an edge f of F into a point. We allow this move only when both endpoints of f lie in Int W and are distinct. The *loop move* adds a small loop based at a vertex of F. The loop should be disjoint from F except at its endpoint and should bound a disk in $W - F$.

4.3 Lemma. *Any two skeletons of W with the same feet can be related by a finite sequence of contraction moves, loop moves, the inverse moves, and isotopies of W constant on ∂W.*

Proof. This lemma is a simple application of the theory of spines of surfaces. A (generalized) *spine* of W is a trivalent graph G embedded in Int W such that $W - G$ consists of a cylinder neighborhood of ∂W and disjoint open 2-disks. Here a graph is *trivalent* if all its vertices have valency 2 or 3. There are three local moves on spines. The *V-move* adds a new vertex of valency 2 inside an edge of the spine. The *H-move* replaces a piece of the spine looking like the letter H by the same piece rotated in the plane of the page to the angle of 90°; see Figure IV.2. The *biangular move* introduces a small biangle (or bigon) in the middle of an edge of the spine, see Figure IV.3. This move replaces a small subarc of the edge with two embedded arcs with the same endpoints. These arcs should be disjoint from the rest of the spine and from ∂W. Note that the inverse V-moves and the inverse biangular moves decrease the number of vertices; we allow these inverse moves only when at least one vertex is left on the component of the spine undergoing the move. We call V-moves, H-moves, biangular moves, and the inverse moves *elementary moves* on spines. It is well known that any two spines of W can be related by a finite sequence of elementary moves and isotopies of W constant on ∂W.

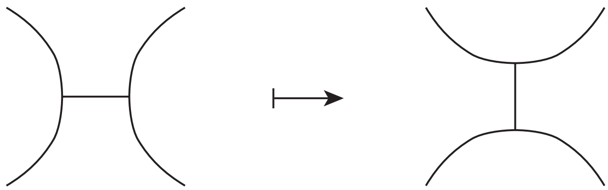


Figure IV.2. The H-move.

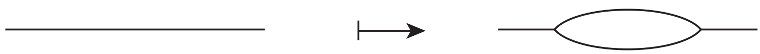


Figure IV.3. The biangular move.

We similarly define H-moves, V-moves, and biangular moves on the skeletons of W. These moves produce new skeletons. As above, the inverse V-moves and the inverse biangular moves on the skeletons of W are allowed only when the component of the skeleton undergoing the move keeps at least one vertex in Int W.

Let us call a skeleton F of W *special* if all its vertices have valency 1, 2, or 3, every leg of F is adjacent to a vertex of valency 3, and F has no vertices adjacent to two or more legs. Any skeleton of W can be transformed into a special one by biangular moves on the legs. Therefore it is enough to prove the lemma for special skeletons.

Deleting from a special skeleton F of W all its legs, we obtain a spine $G = G(F)$ of W. We can reconstruct F from G by adjoining the legs back. These legs lie in the cylinder components of $W - G$ and their feet form the set $F \cap \partial W$. The points of G where the legs are adjoined may vary, but all the skeletons of W obtained in this way can be related by contraction moves and inverse contraction moves. Since any two spines of W can be related by a finite sequence of elementary moves, any two special skeletons of W with the same feet can be related by a sequence of H-moves, V-moves, biangular moves, contraction moves, and the inverse moves. It remains to observe that a V-move on a skeleton is an inverse contraction move, an H-move is a composition of a contraction move with an inverse contraction move, and a biangular move is a composition of a loop move with two inverse contraction moves. □

4.4 Dual CW-decompositions. With each skeleton F of W we associate a *dual regular CW-decomposition* T_F of W. Choose in each component U of $W - F$ a point called its *center*. If U meets ∂W along an arc then the center of U is chosen on this arc. Moreover, if the arc $U \cap \partial W$ contains a base point of ∂W, then we take the latter as the center of U. The centers of the components of $W - F$ are the 0-cells of T_F. Each edge f of F gives rise to a 1-cell $f^* \subset W$ of T_F which crosses f transversely in one point and connects the centers of the components of $W - F$ adjacent to f. If f is a leg of F, then we choose f^* so that $f^* \subset \partial W$ and $f^* \cap f$ is the foot of f. This describes the 0-cells and the 1-cells of T_F. Their union, Γ, is a graph in W containing ∂W. The connected components of $W - \Gamma$ are the 2-cells of T_F. Each of these 2-cells contains a vertex of F lying in Int W. This establishes a bijective correspondence between the 2-cells of T_F and the vertices of F lying in Int W.

By construction, all base points of ∂W are among the vertices of T_F. The CW-decomposition T_F of W is well defined up to isotopy in W constant on the base points of ∂W.

It is easy to see that the formula $F \mapsto T_F$ establishes a bijective correspondence between skeletons of W and regular CW-decomposition of W such that the base points of ∂W are among the vertices. Here both skeletons and CW-decompositions are considered up to isotopies of W constant on the base points of ∂W.

4.5 Moves on CW-decompositions. We define two local moves on a regular CW-decomposition T of W transforming it into another regular CW-decomposition of W. The *edge erasing* deletes an open edge e of T provided e is adjacent to two different faces of T. The resulting regular CW-decomposition T' of W has the same vertices as T and one edge less. The union of e and the two faces of T adjacent to e is a new face of T'. The other faces of T' are the same as in T. The inverse move adds to a regular CW-decomposition a new edge connecting two vertices (possibly equal) inside a face. The *stick erasing* on T deletes a 1-valent vertex of T lying in Int W and deletes the only open edge of T adjacent to this vertex. The inverse move is the stick addition described at the end of Section 3.2, see p. 76. The edge/stick erasings and additions are called *basic moves* on regular CW-decompositions of W.

As an exercise on the basic moves, consider the transformation $T \mapsto T_+$ adding to T one new vertex c inside an open edge e of T. This transformation expands as a composition of basic moves as follows. Let a and b be the endpoints of e (possibly $a = b$). The point c splits e into two half-edges ac and bc. If e is adjacent to two distinct faces of T, then erasing e, we obtain a CW-decomposition T' of W. Consider the stick addition $T' \mapsto T''$ adjoining to T' the vertex c and the edge ac, both lying inside the face of T' created by the move $T \mapsto T'$. Next, add to T'' the edge bc inside the same face. The resulting CW-decomposition of W is isotopic to T_+. If the faces of T adjacent to e from two sides coincide, then we first add to T a new edge e' lying closely to e and having the same vertices a, b. In the resulting CW-decomposition of W, the edge e is adjacent to two distinct faces so that we can apply the previous sequence of basic moves adding c to the set of vertices. Then we erase e' and obtain T_+ as required.

4.6 Lemma. *Any two regular CW-decompositions of W coinciding on ∂W can be related by a finite sequence of basic moves and isotopies of W constant on ∂W.*

Proof. Given two regular CW-decompositions of W coinciding on ∂W, we can present them in the form T_F and $T_{F'}$, where F and F' are skeletons of W with the same feet. By Lemma 4.3, F and F' can be related by a finite sequence of contraction moves, loop moves, the inverse moves, and isotopies of W constant on ∂W. Under the transformation $F \mapsto T_F$, the contraction move and the loop move on the skeletons of W correspond to the edge erasing and the stick addition. Therefore T_F and $T_{F'}$ can be related by a finite sequence of basic moves and isotopies of W constant on ∂W. □

4.7 Proof of Lemmas 3.2.1 and 3.3.1. Lemma 3.2.1 is a special case of Lemma 3.3.1 so that it is enough to prove the latter. Pick a regular CW-decomposition T of W extending the given CW-decomposition of ∂W. It is obvious that an isotopy $T \mapsto T'$ of T in W constant on ∂W transforms any G-system g on T into a G-system g' on T' such that $\langle g \rangle = \langle g' \rangle$. Similarly, any basic move $T \mapsto T'$ lifts to a transformation $g \mapsto g'$ of a G-system g on T into a G-system g' on T' as follows. For an edge/stick erasing, g' is the restriction of g to the edges of T'. For an edge addition, g' is uniquely determined by g and the condition that $g'|_T = g$. For a stick addition, g' is any G-system on T' extending g. In the latter case, g' is not unique since its value on the newly attached edge may be an arbitrary element of G. (Note that g' is unique up to homotopy moves at the newly added vertex.) In all cases, we say that the pair (T', g') is obtained from (T, g) by a basic move. Observe that if g is a characteristic G-system on T, then g' is a characteristic G-system on T' and $\langle g \rangle = \langle g' \rangle$. This was checked for the stick addition at the end of Section 3.2. For the edge erasing, the equality $\langle g \rangle = \langle g' \rangle$ directly follows from formula (1.3.g).

Lemma 4.6 implies that if T and T' are regular CW-decompositions of W, then for any G-system g on T, there is a G-system g' on T' and a sequence of basic moves transforming (T, g) into (T', g'). By the arguments above, $\langle g \rangle = \langle g' \rangle$.

To accomplish the proof, it remains only to show that $\langle g \rangle$ does not depend on the choice of the G-system g on T in its homotopy class. Let g, g' be two characteristic G-systems on T. By the definition of a characteristic system, g and g' coincide on the edges of T lying in ∂W. Also, g is homotopic to g' in the sense that there is a map $\gamma\colon \operatorname{Vert}(T) \to G$ such that $\gamma(\operatorname{Vert}(T_\bullet)) = 1$ and $g'_e = \gamma(i_e) g_e (\gamma(t_e))^{-1}$ for any oriented edge e of T. Here $\operatorname{Vert}(T_\bullet) = T_\bullet$ is the set of base points of ∂W. These conditions imply that γ carries all vertices of T lying in ∂W to 1. An inductive argument shows that it is enough to consider the case where γ takes value 1 on all vertices of T except one vertex $u \in \operatorname{Int} W$. In this case, g' is determined by g and $\gamma(u) \in G$. We shall show that there is a sequence of basic moves transforming (T, g) into (T, g'). This will imply the equality $\langle g \rangle = \langle g' \rangle$.

Suppose first that T does not have edges with both endpoints in u. Let $n \geq 1$ be the valency of u, and let $e_1, \dots, e_n$ be the edges of T adjacent to u, oriented towards u, and numerated cyclically in the order determined by the orientation of W. Let $u_1 u_2 \dots u_n u_1$ be a closed polygon in a small neighborhood of u such that u_i lies on e_i and each side $u_i u_{i+1}$ of the polygon meets $e_1, \dots, e_n$ only at u_i and u_{i+1} (by definition, $u_{n+1} = u_1$). We construct regular CW-decompositions T_1, T_2, T_3 of W as follows. The CW-decomposition T_1 is obtained from T by adding the points $u_1, \dots, u_n$ to the set of vertices. Observe that u_i splits e_i into two oriented subedges, the edge $u_i u$ leading from u_i to u and the complementary edge $e_i^- \subset e_i$ oriented toward u_i. For any $h \in G$, we define a G-system $g *_1 h$ on T_1 by

$$g *_1 h(u_i u) = h, \quad g *_1 h(e_i^-) = g(e_i)h^{-1}, \quad \text{and} \quad g *_1 h(e) = g(e)$$

for all oriented edges e of T not adjacent to u. The construction at the end of Section 4.5 yields a sequence of basic moves transforming (T, g) into $(T_1, g *_1 h)$. Therefore $\langle g \rangle = \langle g *_1 h \rangle$. The CW-decomposition T_2 of W is obtained from T_1 by adding the edges $u_1 u_2$, …, $u_{n-1} u_n$, $u_n u_1$. The G-system $g *_1 h$ on T_1 extends to a G-system $g *_2 h$ on T_2 that takes value 1 on the newly added edges. Deleting the edges $u_1 u$, $u_2 u$, …, $u_{n-1} u$ from T_2, we obtain a CW-decomposition of W having a stick $u_n u$. Deleting this stick, we obtain the CW-decomposition T_3 of W. The G-system $g *_2 h$ restricts to a G-system $g *_3 h$ on T_3. Thus,

$$\langle g \rangle = \langle g *_1 h \rangle = \langle g *_2 h \rangle = \langle g *_3 h \rangle.$$

Observe now that if G-systems g and g' on T are related by a homotopy move at u as above, then $g *_3 h = g' *_3 h'$ for $h = \gamma(u) \in G$ and $h' = 1 \in G$. Therefore

$$\langle g \rangle = \langle g *_3 h \rangle = \langle g' *_3 h' \rangle = \langle g' \rangle.$$

The case where T has edges with both endpoints at u is considered similarly with the only difference that some u_i's may lie on the same edge of T and split it into three subedges.

IV.5 Hermitian biangular G-algebras

In this section we assume that the ground ring K is endowed with an involution $K \to K$, $k \mapsto \bar{k}$. We introduce a class of Hermitian biangular G-algebras and show that their G-centers are Hermitian.

5.1 Anti-involutions. An *anti-involution* on a biangular G-algebra $B = \bigoplus_{\alpha \in G} B_\alpha$ over K is an involutive antilinear antiautomorphism carrying each B_α into $B_{\alpha^{-1}}$. A biangular G-algebra B endowed with an anti-involution is said to be *Hermitian*. The following lemma shows that the G-center of a Hermitian biangular G-algebra over an algebraically closed field is Hermitian. Then, by Theorem III.5.1, the HQFT (A_B, τ_B) is Hermitian.

5.2 Lemma. *Let B be a biangular G-algebra over an algebraically closed field K. Let η be the canonical inner product on B and let $x \mapsto \bar{x}$ be an anti-involution on B. Then*

(i) $\eta(\bar{a}, \bar{b}) = \overline{\eta(a,b)}$ *for all $a, b \in B$;*

(ii) *the G-center L of B is invariant under the anti-involution $x \mapsto \bar{x}$ on B and the restriction of this anti-involution to L turns L into a Hermitian crossed Frobenius G-algebra in the sense of Section* II.7.1.

Proof. Since K is an algebraically closed field, the algebra B_1 splits as a finite direct sum of matrix rings $\mathrm{Mat}_n(K)$ such that n is invertible in K (cf. Section 2.1). This easily implies that for any $c \in B_1$ the trace of the homomorphism $\mu_c \colon B_1 \to B_1$ carrying any $x \in B_1$ to cx is equal to the trace of the homomorphism $B_1 \to B_1$ carrying $x \in B_1$ to xc. To prove the equality $\eta(\bar{a}, \bar{b}) = \overline{\eta(a,b)}$, it suffices to consider the case where $a \in B_\alpha$ and $b \in B_{\alpha^{-1}}$ for some $\alpha \in G$. By definition, $\eta(\bar{a}, \bar{b}) = \mathrm{Tr}(\mu_{\bar{a}\bar{b}} \colon B_1 \to B_1)$. Conjugating $\mu_{\bar{a}\bar{b}} = \mu_{\overline{ba}} \colon B_1 \to B_1$ by the K-antilinear automorphism $x \mapsto \bar{x}$ of B_1, one obtains the K-linear endomorphism $x \mapsto xba$ of B_1. Therefore the trace of the latter endomorphism is equal to $\overline{\mathrm{Tr}\,\mu_{\bar{a}\bar{b}}} = \overline{\eta(\bar{a}, \bar{b})}$. On the other hand, by the argument at the beginning of the proof, this trace is equal to

$$\mathrm{Tr}\,\mu_{ba} = \mathrm{Tr}\,\mu_b \mu_a = \mathrm{Tr}\,\mu_a \mu_b = \mathrm{Tr}\ \mu_{ab} = \eta(a,b).$$

The resulting equality $\overline{\eta(\bar{a}, \bar{b})} = \eta(a,b)$ implies that $\eta(\bar{a}, \bar{b}) = \overline{\eta(a,b)}$.

Recall the vector $\eta_\alpha^- = \sum_i p_i^\alpha \otimes q_i^\alpha \in B_\alpha \otimes B_{\alpha^{-1}}$ for $\alpha \in G$. Claim (i) implies that

$$\sum_i \overline{q_i^\alpha} \otimes \overline{p_i^\alpha} = \sum_i p_i^\alpha \otimes q_i^\alpha.$$

Therefore for all $x \in B_\alpha$,

$$\overline{\psi_\alpha(x)} = \overline{\sum_i p_i^\alpha x q_i^\alpha} = \sum_i \overline{q_i^\alpha}\, \bar{x}\, \overline{p_i^\alpha} = \sum_i p_i^\alpha\, \bar{x}\, q_i^\alpha = \psi_\alpha(\bar{x}).$$

Setting $\alpha = 1$, we obtain $\bar{L} = L$. Moreover, for all $\alpha \in G$, the homomorphism $\varphi_\alpha = \psi_\alpha|_L \colon L \to L$ commutes with the anti-involution $x \mapsto \bar{x}$. Hence, L is a Hermitian crossed Frobenius G-algebra. □

5.3 Unitary biangular G-algebras. Let B be a Hermitian biangular G-algebra over $K = \mathbb{C}$ with complex conjugation in K. Lemma 5.2 yields $\eta(a, \bar{a}) \in \mathbb{R}$ for all $a \in B$.

If $\eta(a, \bar{a}) > 0$ for all non-zero $a \in B$, then B is said to be *unitary*. It is clear that the G-center of a unitary biangular G-algebra is a unitary crossed Frobenius G-algebra. By Corollary III.5.2, the HQFT (A_B, τ_B) is unitary.

5.4 Example. The biangular G-algebra $B = K[G']$ defined in Example 2.3.2 has an anti-involution carrying each element of G' to its inverse. This turns B into a Hermitian biangular G-algebra. The biangular G-algebra B determined by a 2-cocycle in Example 2.3.4 is Hermitian provided the cocycle takes values in the subgroup $\{k \in K \mid k\bar{k} = 1\}$ of K^*. The anti-involution on B is defined by the same formula as in Section II.7.2. If $K = \mathbb{C}$ with complex conjugation, then the biangular G-algebras in both examples are unitary.

Chapter V
Enumeration problems in dimension two

Throughout this chapter, we fix a group epimorphism $q\colon G' \to G$ with finite kernel Γ.

V.1 Enumeration problem for homomorphisms

Consider a homomorphism g from a group π to G. A *lift* of g to G' is a homomorphism $g'\colon \pi \to G'$ such that $qg' = g$. The set (possibly empty) of all such lifts is denoted $\mathrm{Hom}_g(\pi, G')$. If π is finitely generated, then this set is finite. The number of its elements is bounded from above by $|\Gamma|^n$, where n is the minimal number of generators of π. The computation of this number encompasses the problem of finding whether or not g lifts to G'.

The aim of this chapter is to compute the number of elements of $\mathrm{Hom}_g(\pi, G')$ for $\pi = \pi_1(W)$, where W is a closed connected oriented surface. We begin with the case $g = 1$.

1.1 The Frobenius–Mednykh formula. If $g(\pi) = 1$, then, obviously,

$$\mathrm{Hom}_g(\pi, G') = \mathrm{Hom}(\pi, \Gamma).$$

The well-known Frobenius–Mednykh formula says that for $\pi = \pi_1(W)$,

$$|\mathrm{Hom}(\pi,\Gamma)| = |\Gamma| \sum_\rho \big(|\Gamma|/\dim\rho\big)^{-\chi(W)} = |\Gamma| \sum_\rho \big(|\Gamma|/\dim\rho\big)^{2d-2}, \qquad (1.1.a)$$

where the vertical bars stand for the cardinality of a set, d is the genus of W, and ρ runs over the set $\mathrm{Irr}(\Gamma;\mathbb{C})$ of equivalence classes of irreducible finite-dimensional complex representations of Γ. Recall that $|\mathrm{Irr}(\Gamma;\mathbb{C})| = c$, where c is the number of conjugacy classes of elements of Γ. For $W = S^2$, formula (1.1.a) is equivalent to the well-known equality $\sum_\rho (\dim\rho)^2 = |\Gamma|$. For $W = S^1 \times S^1$, formula (1.1.a) was established by Frobenius [Fr]. It may be reformulated in this case by saying that the number of pairs of commuting elements of Γ is equal to $c\,|\Gamma|$. The general case of (1.1.a) is due to Mednykh [Me]. In the context of TQFTs, formula (1.1.a) was rediscovered by Dijkgraaf and Witten [DW] and by Freed and Quinn [FQ].

Note a few simple corollaries of (1.1.a). It is well known that for all $\rho \in \mathrm{Irr}(\Gamma;\mathbb{C})$, the number $\dim\rho$ divides $|\Gamma/Z(\Gamma)| = |\Gamma|/|Z(\Gamma)|$, where $Z(\Gamma)$ denotes the center of Γ. Therefore, if W is a surface of genus $d \geq 1$, then all summands on the right-hand side of (1.1.a) are positive multiples of $|Z(\Gamma)|^{2d-2}$ bounded from above by $|\Gamma|^{2d-2}$. Therefore

$$c\,|\Gamma|^{2d-1} \;\geq\; |\operatorname{Hom}(\pi,\Gamma)| \;\geq\; c\,|\Gamma|\,|Z(\Gamma)|^{2d-2} \tag{1.1.b}$$

and $|\operatorname{Hom}(\pi,\Gamma)|$ is a multiple of $|\Gamma|\,|Z(\Gamma)|^{2d-2}$. If Γ is abelian, then the inequalities (1.1.b) become equalities.

1.2 The general case. To generalize (1.1.a) to arbitrary homomorphisms g from π to G, we recall a few basic notions of the theory of representations. By a (*linear*) *representation* of Γ over K, we mean a group homomorphism $\Gamma \to \mathrm{GL}_n(K)$, where $n = 1, 2, \ldots$. Here $\mathrm{GL}_n(K)$ is the group of invertible $(n \times n)$-matrices over K. Two representations $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ and $\rho'\colon \Gamma \to \mathrm{GL}_{n'}(K)$ are *equivalent* if $n = n'$ and there is a matrix $M \in \mathrm{GL}_n(K)$ such that $\rho'(h) = M^{-1}\rho(h)M$ for all $h \in \Gamma$. A representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is *irreducible* if the induced action of Γ on K^n preserves no K-submodule of K^n except 0 and K^n. The set of equivalence classes of irreducible representations of Γ over K is denoted $\mathrm{Irr}(\Gamma; K)$.

Given $a \in G'$ and an irreducible representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$, the formula $h \mapsto \rho(a^{-1}ha)\colon \Gamma \to \mathrm{GL}_n(K)$ defines an irreducible representation of Γ denoted $a\rho$. This defines a left action of G' on $\mathrm{Irr}(\Gamma)$. The induced action of $\Gamma \subset G'$ is trivial because $a\rho = \rho(a)^{-1}\,\rho\,\rho(a)$ for any $a \in \Gamma$. We obtain thus a left action of $G = G'/\Gamma$ on $\mathrm{Irr}(\Gamma; K)$. The stabilizer $G_\rho \subset G$ of an irreducible representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ consists of all $\alpha \in G$ such that for some (and then for any) $a \in q^{-1}(\alpha)$ the representations ρ and $a\rho$ are equivalent. It is clear that G_ρ is a subgroup of G depending only on the equivalence class of ρ.

Under certain assumptions on ρ, we associate with ρ a cohomology class $\zeta_\rho \in H^2(G_\rho; K^*)$ depending only on the equivalence class of ρ; see Section 2. The assumptions in question are fulfilled for all irreducible ρ provided K is an algebraically closed field of characteristic zero.

We now state the main theorem of this chapter.

1.2.1 Theorem. *Let W be a closed connected oriented surface with fundamental group π. Let $g\colon \pi \to G$ be a group homomorphism and let K be an algebraically closed field of characteristic zero. Then*

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma| \sum_{\substack{\rho \in \mathrm{Irr}(\Gamma; K)\\ G_\rho \supset g(\pi)}} \big(|\Gamma|/\dim\rho\big)^{-\chi(W)}\, g^*(\zeta_\rho)([W]), \tag{1.2.a}$$

where ρ runs over the equivalence classes of irreducible representations of Γ over K such that $G_\rho \supset g(\pi)$, and $g^*(\zeta_\rho)([W]) \in K^*$ is the evaluation of $g^*(\zeta_\rho) \in H^2(\pi; K^*)$ on the fundamental class $[W] \in H_2(W;\mathbb{Z}) = H_2(\pi;\mathbb{Z})$ of W.

Here the cohomology class $g^*(\zeta_\rho)$ is well defined because $g(\pi) \subset G_\rho$. The equality $H_2(W;\mathbb{Z}) = H_2(\pi;\mathbb{Z})$ follows from the fact that W is an Eilenberg–MacLane space. The evaluation of elements of $H^2(\pi; K^*)$ on elements of $H_2(\pi;\mathbb{Z})$ is induced by the bilinear form $K^* \times \mathbb{Z} \to K^*$, $(k,n) \mapsto k^n$. The summation on the right-hand side of (1.2.a) is the summation in K.

The sum on the right-hand side of (1.2.a) is always non-empty because the trivial one-dimensional representation $\rho_0 \colon \Gamma \to \{1\} \in K^* = \mathrm{GL}_1(K)$ satisfies $G_{\rho_0} = G \supset g(\pi)$. This representation contributes the summand $|\Gamma|^{-\chi(W)}$ since $g^*(\zeta_{\rho_0})([W]) = 1$.

Note a few interesting special cases of (1.2.a). If $g \colon \pi \to G$ is an epimorphism, then the inclusion $G_\rho \supset g(\pi)$ simplifies to $G_\rho = G$. In this case the set of ρ such that $G_\rho \supset g(\pi)$ is nothing but the fixed point set of the action of G on $\mathrm{Irr}(\Gamma; K)$.

If $g = 1$, then $G_\rho \supset g(\pi)$ and $g^*(\zeta_\rho)([W]) = 1$ for all ρ. Formula (1.2.a) is then equivalent to (1.1.a).

If $G' = \Gamma \times G$ and $q \colon G' \to G$ is the projection, then (1.2.a) directly follows from (1.1.a) since in this case $\mathrm{Hom}_g(\pi, G') = \mathrm{Hom}(\pi, \Gamma)$ and ζ_ρ is trivial for all ρ.

If $\Gamma = \mathrm{Ker}\, q$ is central in G', then formula (1.2.a) can be obtained by a rather simple algebraic argument. In this case $G_\rho = G$ and $\dim \rho = 1$ for all ρ, while ζ_ρ is the image of the standard cohomology class $\zeta \in H^2(G; \Gamma)$ determined by q under the coefficient homomorphism $H^2(G; \Gamma) \to H^2(G; K^*)$ induced by $\rho \colon \Gamma \to K^*$. Formula (1.2.a) can be deduced then from the following two easy assertions: g has a lift to G' if and only if $g^*(\zeta) = 1$; if there are such lifts, then their number is equal to $|\Gamma|^{2-\chi(W)}$.

Theorem 1.2.1 will be generalized and proved in Section 6. The proof uses the theory of 2-dimensional HQFTs developed above. Note that formula (1.1.a) admits a direct algebraic proof; see [Jo]. It would be interesting to give a purely algebraic proof of Theorem 1.2.1.

Theorem 1.2.1 was announced in [Tu6] and proved in [Tu7]. This theorem extends to orientable surfaces with boundary (see [Tu7]) and to non-orientable surfaces. Formula (1.1.a) extends to homomorphisms from $\pi = \pi_1(W)$ to certain Lie groups; see [Wi2]. Theorem 1.2.1 presumably admits similar extensions.

1.3 Corollaries. We keep the assumptions of Theorem 1.2.1 and establish several corollaries of this theorem.

1.3.1 Corollary. *The homomorphism $g \colon \pi = \pi_1(W) \to G$ lifts to G' if and only if*

$$\sum_{\substack{\rho \in \mathrm{Irr}(\Gamma; K) \\ G_\rho \supset g(\pi)}} (\dim \rho)^{\chi(W)}\, g^*(\zeta_\rho)([W]) \neq 0.$$

Observe that

$$g^*(\zeta_\rho)([W]) = (\zeta_\rho|_{g(\pi)})(g_*([W])),$$

where $\zeta_\rho|_{g(\pi)} \in H^2(g(\pi); K^*)$ is the restriction of ζ_ρ to $g(\pi) \subset G_\rho$ and

$$g_* \colon H_2(W; \mathbb{Z}) = H_2(\pi; \mathbb{Z}) \to H_2(g(\pi); \mathbb{Z})$$

is the homomorphism induced by g. Formula (1.2.a) can be rewritten as

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma| \sum_{\substack{\rho\in\operatorname{Irr}(\Gamma;K)\\ G_\rho\supset g(\pi)}} \big(|\Gamma|/\dim\rho\big)^{-\chi(W)} (\zeta_\rho|_{g(\pi)})(g_*([W])). \tag{1.3.a}$$

1.3.2 Corollary. *The number* $|\operatorname{Hom}_g(\pi, G')|$ *is entirely determined by the homomorphism* $q\colon G' \to G$*, the genus of* W*, the group* $g(\pi) \subset G$*, and the homology class* $g_*([W]) \in H_2(g(\pi); \mathbb{Z})$*.*

For example, if $g(\pi) = G$ and $g_*([W]) = 0 \in H_2(G)$, then (1.3.a) implies that $|\operatorname{Hom}_g(\pi, G')|$ is a positive integer divisible by $|\Gamma|$. Thus, any epimorphism $g\colon \pi \to G$ with $g_*([W]) = 0$ has a lift to G', and the number of such lifts is greater than or equal to $|\Gamma|$.

The following assertion will be proven in Section 7.

1.3.3 Corollary. *If* $W \neq S^2$*, then for any homomorphism* $g\colon \pi = \pi_1(W) \to G$*, the number* $|\operatorname{Hom}_g(\pi, G')|$ *is divisible by* $|\Gamma|\,|Z(\Gamma)|^{2d-2}$*, where* $d \geq 1$ *is the genus of* W *and* $Z(\Gamma)$ *is the center of* Γ*.*

If $K = \mathbb{C}$, then for all ρ in Theorem 1.2.1,

$$\zeta_\rho \in H^2(G_\rho; S^1) \subset H^2(G_\rho; \mathbb{C}^*); \tag{1.3.b}$$

see Section 2. Therefore $g^*(\zeta_\rho)([W]) \in S^1$ and

$$|\, g^*(\zeta_\rho)([W])\,| = 1. \tag{1.3.c}$$

The formulas (1.1.a), (1.2.a), and (1.3.c) imply the following assertion.

1.3.4 Corollary. *For any homomorphism* $g\colon \pi = \pi_1(W) \to G$*,*

$$|\operatorname{Hom}_g(\pi, G')| \leq |\operatorname{Hom}(\pi, \Gamma)|. \tag{1.3.d}$$

This inequality is an equality if and only if $g(\pi) \subset G_\rho$ *and* $g^*(\zeta_\rho)([W]) = 1$ *for all irreducible (finite-dimensional) complex representations* ρ *of* Γ*.*

The inequality (1.3.d) does not hold for arbitrary groups π. For example, let G' be the group of permutations of the set $\{1, 2, 3\}$ and let $q\colon G' \to \mathbb{Z}/2\mathbb{Z}$ be the surjection carrying all transpositions to 1 (mod 2). Clearly, $\Gamma = \operatorname{Ker} q = \mathbb{Z}/3\mathbb{Z}$. Let π be the group with $m \geq 1$ generators $x_1, \ldots, x_m$ and defining relations $x_1^2 = x_2^2 = \cdots = x_m^2$. A homomorphism $\pi \to \Gamma$ has to carry all the generators $x_1, \ldots, x_m$ to the same element. Therefore, $|\operatorname{Hom}(\pi, \Gamma)| = 3$. On the other hand, the homomorphism $\pi \to \mathbf{Z}/2\mathbb{Z}$ carrying $x_1, \ldots, x_m$ to 1 (mod 2) admits at least 3^m lifts to G' carrying $x_1, \ldots, x_m$ to arbitrary transpositions.

A *section* of a group homomorphism $p\colon \pi' \to \pi$ is a homomorphism $s\colon \pi \to \pi'$ such that $ps = \mathrm{id}_\pi$. The set of sections of p (possibly empty) is denoted $S_*(p)$.

1.3.5 Corollary. *Let $p\colon \pi' \to \pi = \pi_1(W)$ be a group epimorphism with finite kernel Φ. Then*

$$|S_*(p)| = |\Phi| \sum_{\substack{\rho \in \mathrm{Irr}(\Phi;K)\\ \pi_\rho = \pi}} (|\Phi|/\dim\rho)^{-\chi(W)}\, \zeta_\rho([W]),$$

where $\zeta_\rho([W]) \in K^*$ is the evaluation of $\zeta_\rho \in H^2(\pi;K^*)$ on $[W] \in H_2(\pi;\mathbb{Z})$.

This is obtained by setting in Theorem 1.2.1, $G = \pi$, $G' = \pi'$, $\Gamma = \Phi$, $q = p$, and $g = \mathrm{id}\colon \pi \to \pi$. Corollary 1.3.4 implies that

$$|S_*(p)| \le |\operatorname{Hom}(\pi, \Phi)|.$$

This inequality is an equality if and only if $\pi_\rho = \pi$ and $\zeta_\rho([W]) = 1$ for all irreducible complex representations ρ of Φ. Corollary 1.3.3 implies that if $W \neq S^2$, then the number $|S_*(p)|$ is divisible by $|\Phi|\,|Z(\Phi)|^{2d-2}$, where $d \ge 1$ is the genus of W.

Further corollaries of Theorem 1.2.1 will be discussed in Sections 7–9.

V.2 Linear representations of Γ and cohomology

Fix a representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ with $n \ge 1$. We define the cohomology class $\zeta_\rho \in H^2(G_\rho; K^*)$ appearing in Theorem 1.2.1. We first introduce a cohomology class $\kappa_\rho \in H^2(G_\rho; K^*)$ which is somewhat weaker than ζ_ρ but is defined for all ρ. Then, under an irreducibility-type assumption on ρ, we define ζ_ρ.

2.1 The cohomology class κ_ρ. Recall the group $G_\rho \subset G$ from Section 1.2. For each $\alpha \in G_\rho$, pick $\tilde\alpha \in q^{-1}(\alpha) \subset G'$. We assume that $\tilde 1 = 1 \in G'$. For any $\alpha, \beta \in G_\rho$, set $h_{\alpha,\beta} = \widetilde{\alpha\beta}^{-1}\,\tilde\alpha\,\tilde\beta \in \Gamma$ and

$$\kappa_{\alpha,\beta} = \det \rho(h_{\alpha,\beta}) \in K^*.$$

The following lemma shows that

$$\{\kappa_{\alpha,\beta}\}_{\alpha,\beta \in G_\rho} \tag{2.1.a}$$

is a normalized 2-cocycle on G_ρ. Its class in $H^2(G_\rho; K^*)$ is denoted by κ_ρ. It is clear that κ_ρ is an obstruction to the existence of a homomorphism $s\colon G_\rho \to G'$ such that $qs = \mathrm{id}$.

2.1.1 Lemma. *The family (2.1.a) is a normalized 2-cocycle on G_ρ. Its cohomology class κ_ρ does not depend on the choice of the lifts $\{\tilde\alpha\}_{\alpha \in G_\rho}$.*

Proof. It is clear that $\kappa_{1,1} = 1$. For any $\alpha, \beta, \gamma \in G_\rho$,

$$\tilde\alpha\,\tilde\beta\,\tilde\gamma = \widetilde{\alpha\beta}\, h_{\alpha,\beta}\,\tilde\gamma = \widetilde{\alpha\beta}\,\tilde\gamma\,\tilde\gamma^{-1}\, h_{\alpha,\beta}\,\tilde\gamma = \widetilde{\alpha\beta\gamma}\, h_{\alpha\beta,\gamma}\,\tilde\gamma^{-1}\, h_{\alpha,\beta}\,\tilde\gamma.$$

On the other hand,

$$\tilde\alpha\,\tilde\beta\,\tilde\gamma = \tilde\alpha\,\widetilde{\beta\gamma}\,h_{\beta,\gamma} = \widetilde{\alpha\beta\gamma}\,h_{\alpha,\beta\gamma}\,h_{\beta,\gamma}.$$

Therefore

$$h_{\alpha\beta,\gamma}\,\tilde\gamma^{-1}\,h_{\alpha,\beta}\,\tilde\gamma = h_{\alpha,\beta\gamma}\,h_{\beta,\gamma}.$$

Applying ρ, we obtain

$$\rho(h_{\alpha\beta,\gamma})\,\rho(\tilde\gamma^{-1}\,h_{\alpha,\beta}\,\tilde\gamma) = \rho(h_{\alpha,\beta\gamma})\,\rho(h_{\beta,\gamma}).$$

The inclusion $\gamma \in G_\rho$ implies that the matrix $\rho(\tilde\gamma^{-1}\,h_{\alpha,\beta}\,\tilde\gamma)$ is conjugate to $\rho(h_{\alpha,\beta})$. Taking the determinant we obtain $\kappa_{\alpha\beta,\gamma}\,\kappa_{\alpha,\beta} = \kappa_{\alpha,\beta\gamma}\,\kappa_{\beta,\gamma}$.

To prove the second claim of the lemma, suppose that each $\tilde\alpha$ is replaced by $\tilde\alpha' = \tilde\alpha\,g_\alpha$ with $g_\alpha \in \Gamma$. For $\alpha, \beta \in G_\rho$, we have $\tilde\alpha'\,\tilde\beta' = \widetilde{\alpha\beta}'\,h'_{\alpha,\beta}$, where

$$h'_{\alpha,\beta} = g_{\alpha\beta}^{-1}\,h_{\alpha,\beta}\,\tilde\beta^{-1}\,g_\alpha\,\tilde\beta\,g_\beta. \tag{2.1.b}$$

Then

$$\rho(h'_{\alpha,\beta}) = \rho(g_{\alpha\beta}^{-1})\,\rho(h_{\alpha,\beta})\,\rho(\tilde\beta^{-1}\,g_\alpha\,\tilde\beta)\,\rho(g_\beta).$$

Taking the determinant and using the inclusion $\beta \in G_\rho$, we obtain

$$\det\rho(h'_{\alpha,\beta}) = \det\,\rho(g_\alpha)\ \det\rho(g_\beta)\ \det\rho(g_{\alpha\beta})^{-1}\,\kappa_{\alpha,\beta}.$$

This shows that the cohomology class of the cocycle $\{\kappa_{\alpha,\beta}\}_{\alpha,\beta}$ does not depend on the choice of the lifts $\{\tilde\alpha\}_\alpha$. □

2.2 The cohomology class ζ_ρ. Assume that $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is a *Schur representation* in the sense that the only matrices $A \in \mathrm{GL}_n(K)$ such that $A\,\rho(h) = \rho(h)\,A$ for all $h \in \Gamma$, are scalar matrices. We define $\zeta_\rho \in H^2(G_\rho; K^*)$ as follows.

As above, choose $\tilde\alpha \in q^{-1}(\alpha)$ for each $\alpha \in G_\rho$ so that $\tilde 1 = 1$. By definition of G_ρ, for $\alpha \in G_\rho$, the representation $\tilde\alpha\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is equivalent to ρ. Thus, there is a matrix $M_\alpha \in \mathrm{GL}_n(K)$ such that for all $h \in \Gamma$,

$$\rho(\tilde\alpha^{-1}\,h\,\tilde\alpha) = M_\alpha^{-1}\,\rho(h)\,M_\alpha. \tag{2.2.a}$$

By the assumptions on ρ, such a matrix M_α is unique up to multiplication by an element of K^*; we fix M_α for all $\alpha \in G_\rho$. For $\alpha = 1$, we take M_α to be the unit $(n \times n)$-matrix. As above, set $h_{\alpha,\beta} = \widetilde{\alpha\beta}^{-1}\,\tilde\alpha\,\tilde\beta\,\in \Gamma$ for $\alpha, \beta \in G_\rho$.

2.2.1 Lemma. *For any $\alpha, \beta \in G_\rho$, there is a unique $\zeta_{\alpha,\beta} \in K^*$ such that*

$$\zeta_{\alpha,\beta}\,M_\alpha\,M_\beta = M_{\alpha\beta}\,\rho(h_{\alpha,\beta}). \tag{2.2.b}$$

The family $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is a normalized 2-cocycle on G_ρ. Its cohomology class $\zeta_\rho \in H^2(G_\rho; K^*)$ depends only on the equivalence class of ρ and does not depend on the choice of the matrices $\{M_\alpha\}_\alpha$ or the lifts $\{\tilde\alpha\}_\alpha$.

Proof. For $\alpha, \beta \in G_\rho$ and $h \in \Gamma$,

$$\begin{aligned} M_{\alpha\beta}^{-1}\, \rho(h)\, M_{\alpha\beta} &= \rho(\widetilde{\alpha\beta}^{-1}\, h\, \widetilde{\alpha\beta}) \\ &= \rho(h_{\alpha,\beta}\, \tilde\beta^{-1}\, \tilde\alpha^{-1}\, h\, \tilde\alpha\, \tilde\beta\, h_{\alpha,\beta}^{-1}) \\ &= \rho(h_{\alpha,\beta})\, M_\beta^{-1}\, M_\alpha^{-1}\, \rho(h)\, M_\alpha\, M_\beta\, \rho(h_{\alpha,\beta})^{-1}. \end{aligned}$$

Since ρ is a Schur representation, there is a unique $\zeta_{\alpha,\beta} \in K^*$ satisfying (2.2.b).

Consider the subgroup $G'_\rho = q^{-1}(G_\rho)$ of G'. This subgroup contains Γ and any element of G'_ρ expands uniquely in the form $\tilde\alpha g$ with $\alpha \in G_\rho$ and $g \in \Gamma$. The formula

$$\rho_+(\tilde\alpha g) = M_\alpha\, \rho(g) \tag{2.2.c}$$

defines a mapping $\rho_+ \colon G'_\rho \to \mathrm{GL}_n(K)$ extending $\rho\colon \Gamma \to \mathrm{GL}_n(K)$. By definition, $\rho_+(\tilde\alpha) = M_\alpha$ for all $\alpha \in G_\rho$. Observe that for all $a \in G'_\rho$ and $h \in \Gamma$,

$$\rho_+(a)\, \rho(h) = \rho_+(a\, h). \tag{2.2.d}$$

To see this, expand $a = \tilde\alpha g$, where $\alpha = q(a) \in G_\rho$ and $g \in \Gamma$. Then

$$\rho_+(a)\, \rho(h) = M_\alpha\, \rho(g)\, \rho(h) = M_\alpha\, \rho(g\, h) = \rho_+(\tilde\alpha\, g\, h) = \rho_+(a\, h).$$

Observe also that for $\alpha, \beta \in G_\rho$,

$$\begin{aligned} \rho_+(\tilde\alpha\, \tilde\beta) &= \rho_+(\widetilde{\alpha\beta}\, h_{\alpha,\beta}) = M_{\alpha\beta}\, \rho(h_{\alpha,\beta}) \\ &= \zeta_{\alpha,\beta}\, M_\alpha\, M_\beta = \zeta_{\alpha,\beta}\, \rho_+(\tilde\alpha)\, \rho_+(\tilde\beta). \end{aligned} \tag{2.2.e}$$

We claim that more generally, for all $a, b \in G'_\rho$,

$$\rho_+(ab) = \zeta_{q(a),q(b)}\, \rho_+(a)\, \rho_+(b). \tag{2.2.f}$$

Indeed, let $a = \tilde\alpha g$, $b = \tilde\beta h$ with $\alpha, \beta \in G_\rho$ and $g, h \in \Gamma$. Then

$$\begin{aligned} \rho_+(a\, b) = \rho_+(\tilde\alpha\, g\, \tilde\beta\, h) &= \rho_+(\tilde\alpha\, \tilde\beta\, \tilde\beta^{-1}\, g\, \tilde\beta\, h) \\ &= \rho_+(\tilde\alpha\, \tilde\beta)\, \rho(\tilde\beta^{-1}\, g\, \tilde\beta)\, \rho(h) \\ &= \zeta_{\alpha,\beta}\, M_\alpha\, M_\beta\, \rho(\tilde\beta^{-1}\, g\, \tilde\beta)\, \rho(h) \\ &= \zeta_{\alpha,\beta}\, M_\alpha\, M_\beta\, M_\beta^{-1}\, \rho(g)\, M_\beta\, \rho(h) \\ &= \zeta_{\alpha,\beta}\, M_\alpha\, \rho(g)\, M_\beta\, \rho(h) = \zeta_{\alpha,\beta}\, \rho_+(a)\, \rho_+(b), \end{aligned}$$

where we use formulas (2.2.a), (2.2.c)–(2.2.e).

Formula (2.2.f) and the associativity of multiplication in G'_ρ imply that the family $\{\zeta_{q(a),q(b)}\}_{a,b}$ is a 2-cocycle on G'_ρ. Therefore, the family $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is a 2-cocycle on G_ρ. It is normalized, since $\zeta_{1,1} = 1$. That the cohomology class of this 2-cocycle does not depend on the choice of $\{M_\alpha\}_\alpha$ follows directly from (2.2.b). If $\tilde\alpha$ is traded for

$\tilde{\alpha}' = \tilde{\alpha}\, g_\alpha$ with $g_\alpha \in \Gamma$, then $h_{\alpha,\beta}$ is traded for $h'_{\alpha,\beta}$ given by (2.1.b) and M_α is traded for $M'_\alpha = M_\alpha\, \rho(g_\alpha)$. Substituting these new values of h and M in (2.2.b), we obtain the same cocycle $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$.

When $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is conjugated by $M \in \mathrm{GL}_n(K)$, the group G_ρ is preserved, while the matrices $\{M_\alpha\}_\alpha$ are replaced by $\{M^{-1} M_\alpha M\}_\alpha$. Formula (2.2.b) shows that the cocycle $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is then preserved. Thus, ζ_ρ depends only on the equivalence class of ρ. □

2.3 Properties of κ_ρ and ζ_ρ. The following two lemmas show that ζ_ρ determines κ_ρ and both κ_ρ and ζ_ρ have finite order in $H^2(G_\rho; K^*)$. Note that we use additive notation for the group operation in $H^2(G_\rho; K^*)$ induced by multiplication in K^*.

2.3.1 Lemma. *We have $\kappa_\rho = n\, \zeta_\rho$, where $n = \dim \rho$.*

Proof. This lemma is a direct consequence of (2.2.b): taking the determinant on both sides we obtain that the cocycle $\zeta^n_{\alpha,\beta}$ representing $n\,\zeta_\rho$ differs from the cocycle $\det \rho(h_{\alpha,\beta})$ representing κ_ρ by a coboundary. □

2.3.2 Lemma. *Let a_ρ be the minimal positive integer a such that* $\det \rho(h)^a = 1$ *for all $h \in \Gamma$. Then $a_\rho\, \kappa_\rho = 0$ and $a_\rho\, (\dim \rho)\, \zeta_\rho = 0$.*

Proof. The integer a_ρ exists because the homomorphism $\Gamma \to K^*, h \mapsto \det \rho(h)$ splits as a composition of the projection from Γ to the finite abelian group $H_1(\Gamma) = \Gamma/[\Gamma,\Gamma]$ and a homomorphism $H_1(\Gamma) \to K^*$. The equality $a_\rho\, \kappa_\rho = 0$ follows from the definitions. By the previous lemma, $a_\rho\, (\dim \rho)\, \zeta_\rho = a_\rho\, \kappa_\rho = 0$. □

2.3.3 Lemma. *Set $G'_\rho = q^{-1}(G_\rho)$. Let $q^*\colon H^2(G_\rho; K^*) \to H^2(G'_\rho; K^*)$ be the homomorphism induced by q. Then*

$$(\dim \rho)\, q^*(\zeta_\rho) = q^*(\kappa_\rho) = 0. \tag{2.3.a}$$

Proof. We use notation of Section 2.1. By Lemma 2.3.1, it is enough to prove that $q^*(\kappa_\rho) = 0$. By definition, the cohomology class $q^*(\kappa_\rho) \in H^2(G'_\rho; K^*)$ is represented by the cocycle

$$\{\det \rho(h_{q(a),q(b)})\}_{a,b \in G'_\rho}. \tag{2.3.b}$$

For any $a \in G'_\rho$, there is a unique $g_a \in \Gamma$ such that $\widetilde{q(a)} = a\, g_a$. Then

$$h_{q(a),q(b)} = \widetilde{q(a)\, q(b)}^{-1}\, \widetilde{q(a)}\, \widetilde{q(b)} = \widetilde{q(ab)}^{-1}\, \widetilde{q(a)}\, \widetilde{q(b)} = g_{ab}^{-1}\, (b^{-1} g_a b)\, g_b.$$

Applying ρ and taking the determinant, we obtain

$$\det \rho(h_{q(a),q(b)}) = \det \rho(g_{ab})^{-1}\, \det \rho(g_a)\, \det \rho(g_b).$$

Hence, the cocycle (2.3.b) is a coboundary. So, $q^*(\kappa_\rho) = 0$. □

Further properties of κ_ρ and ζ_ρ will be studied in Section 4 in a more general setting of projective representations of Γ.

2.4 Proof of Formula (1.3.b). In (1.3.b) it is assumed that $K = \mathbb{C}$ and ρ is an irreducible representation of Γ. Since Γ is finite, the image of the homomorphism $\Gamma \to \mathbb{C}^*, h \mapsto \det \rho(h)$ is a finite subgroup of $\mathbb{C}^*$. Such a subgroup is contained in the unit circle S^1. Therefore $\det \rho(h) \in S^1$ for all $h \in \Gamma$. Multiplying all the matrices M_α in the definition of ζ_ρ by non-zero complex numbers, we can ensure that $\det M_\alpha = 1$ for all $\alpha \in G_\rho$. Formula (2.2.b) implies then that $\zeta_{\alpha,\beta} \in S^1$ for all $\alpha, \beta \in G_\rho$. This yields (1.3.b).

2.5 Remark. The definition and properties of κ_ρ and ζ_ρ (except Lemma 2.3.2) do not use the finiteness of $\Gamma = \operatorname{Ker}(q\colon G' \to G)$ and directly extend to group epimorphisms with infinite kernel. The same is true for the definitions and results of Sections 3 and 4.

V.3 Projective representations of Γ

The constructions of Section 2 are generalized here to projective representations of the group $\Gamma = \operatorname{Ker} q$ associated with a certain 2-cocycle θ on G'. This will be instrumental in the generalization of Theorem 1.2.1 in Section 6. The reader interested mainly in the proof of Theorem 1.2.1 may skip this section and assume everywhere in the sequel that $\theta = 1$.

Fix throughout this section a normalized 2-cocycle $\theta = \{\theta_{a,b} \in K^*\}_{a,b \in G'}$.

3.1 θ-representations of Γ. A mapping $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ with $n = 1, 2, \dots$ is a *θ-representation of* Γ if $\rho(1)$ is the unit $n \times n$ matrix and $\rho(g)\,\rho(h) = \theta_{g,h}\,\rho(gh)$ for all $g, h \in \Gamma$. For a θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ and a matrix $M \in \mathrm{GL}_n(K)$, the mapping $M^{-1}\rho M\colon \Gamma \to \mathrm{GL}_n(K)$ carrying $h \in \Gamma$ to $M^{-1}\rho(h)M$, is a θ-representation. We say that two θ-representations $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ and $\rho'\colon \Gamma \to \mathrm{GL}_{n'}(K)$ are *equivalent* and write $\rho \sim \rho'$ if $n = n'$ and $\rho' = M^{-1}\rho M$ for some $M \in \mathrm{GL}_n(K)$. Clearly, $\sim$ is an equivalence relation on the set of θ-representations of Γ. We denote by $\mathcal{R}_\theta$ the corresponding set of equivalence classes. For more on projective representations of groups; see [Kar].

The definition of θ-representations of Γ uses only the restriction of the cocycle θ to Γ. We now use the other values of θ to define an action of G' on $\mathcal{R}_\theta$. Given $a \in G'$ and a θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$, consider a mapping $a\rho\colon \Gamma \to \mathrm{GL}_n(K)$, whose value on any $h \in \Gamma$ is given by

$$a\rho(h) = \theta^{-1}_{a,a^{-1}}\,\theta_{a^{-1},ha}\,\theta_{h,a}\,\rho(a^{-1}ha). \tag{3.1.a}$$

3.1.1 Lemma. *The mapping $a\rho$ is a θ-representation of Γ. The formula $(a, \rho) \mapsto a\rho$ defines a left action of G' on $\mathcal{R}_\theta$. This action induces a left action of G on $\mathcal{R}_\theta$.*

Proof. Consider the associative algebra $B = \bigoplus_{a \in G'} K l_a$ with multiplication $l_a\, l_b = \theta_{a,b}\, l_{ab}$ for $a, b \in G'$ and with unit l_1, where 1 is the neutral element of G'. It is clear that $l_a \in B$ is invertible in B for all $a \in G'$; in fact, $l_a^{-1} = \theta_{a,a^{-1}}^{-1} l_{a^{-1}}$. Since Γ is a normal subgroup of G', the module $B_1 = \bigoplus_{h\in\Gamma} Kl_h$ is a subalgebra of B such that $l_a^{-1} B_1 l_a \subset B_1$ for all $a \in G'$.

Any mapping $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ induces a K-linear homomorphism

$$\bar{\rho}\colon B_1 \to \mathrm{Mat}_n(K)$$

by $\bar{\rho}(l_h) = \rho(h)$ for $h \in \Gamma$. It is clear that ρ is a θ-representation if and only if $\bar{\rho}$ is an algebra homomorphism.

Given a θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ and $a \in G'$, we can rewrite the definition of $a\rho\colon \Gamma \to \mathrm{GL}_n(K)$ in the form

$$a\rho(h) = \bar{\rho}(l_a^{-1}\, l_h\, l_a)$$

for all $h \in \Gamma$. Then $\overline{a\rho}(l) = \bar{\rho}(l_a^{-1}\, l\, l_a)$ for all $l \in B_1$. Since $\bar{\rho}$ is an algebra homomorphism, so is $\overline{a\rho}\colon B_1 \to \mathrm{Mat}_n(K)$ and therefore $a\rho$ is a θ-representation of Γ.

It is clear that if $\rho \sim \rho'$, then $a\rho \sim a\rho'$. For any $a, b \in G'$ and $h \in \Gamma$,

$$\begin{aligned}(ab)\rho(h) &= \bar{\rho}(l_{ab}^{-1}\, l_h\, l_{ab})\\ &= \bar{\rho}((\theta_{a,b}^{-1}\, l_a\, l_b)^{-1}\, l_h\, \theta_{a,b}^{-1}\, l_a\, l_b)\\ &= \bar{\rho}(l_b^{-1}\, l_a^{-1}\, l_h\, l_a\, l_b)\\ &= a(b\rho)(h).\end{aligned}$$

This and the obvious equality $1\rho = \rho$ prove the second claim of the lemma.

To prove the last claim of the lemma, we must show that the action of $\Gamma \subset G'$ on $\mathcal{R}_\theta$ is trivial. If $a \in \Gamma$, then

$$a\rho(h) = \bar{\rho}(l_a^{-1}\, l_h\, l_a) = \bar{\rho}(l_a)^{-1}\, \bar{\rho}(l_h)\, \bar{\rho}(l_a) = \rho(a)^{-1}\, \rho(h)\, \rho(a)$$

for all $h \in \Gamma$. Therefore $a\rho \sim \rho$. □

3.2 The cohomology class κ_ρ. Let $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ be a θ-representation of Γ. Let $G_\rho \subset G$ be the stabilizer of ρ, i.e., the subgroup of G consisting of all $\alpha \in G$ such that $\alpha[\rho] = [\rho]$, where $[\rho] \in \mathcal{R}_\theta$ is the equivalence class of ρ. By definition, the group G_ρ depends only on $[\rho]$. An element α of G belongs to G_ρ if and only if $a\rho \sim \rho$ for some (and then for all) $a \in q^{-1}(\alpha) \subset G'$.

We define a cohomology class $\kappa_\rho \in H^2(G_\rho; K^*)$ as follows. Fix for each $\alpha \in G_\rho$ an element $\tilde{\alpha}$ of $q^{-1}(\alpha)$ and assume that $\tilde{1} = 1 \in G'$. For $\alpha, \beta \in G_\rho$, set $h_{\alpha,\beta} = \widetilde{\alpha\beta}^{-1}\, \tilde{\alpha}\, \tilde{\beta} \in \Gamma$ and

$$\kappa_{\alpha,\beta} = (\theta_{\tilde{\alpha},\tilde{\beta}}\, \theta^{-1}_{\widetilde{\alpha\beta},h_{\alpha,\beta}})^n \det \rho(h_{\alpha,\beta}) \in K^*. \tag{3.2.a}$$

This formula can be rewritten in terms of the algebras B, $B_1 \subset B$ and the algebra homomorphism $\bar{\rho}\colon B_1 \to \mathrm{Mat}_n(K)$ introduced in the proof of Lemma 3.1.1. Set

$$L_{\alpha,\beta} = \theta_{\tilde{\alpha},\tilde{\beta}}\, \theta^{-1}_{\widetilde{\alpha\beta},h_{\alpha,\beta}}\, l_{h_{\alpha,\beta}} \in B_1. \tag{3.2.b}$$

Then

$$\kappa_{\alpha,\beta} = \det\, \bar{\rho}(L_{\alpha,\beta}).$$

3.2.1 Lemma. *The family $\{\kappa_{\alpha,\beta}\}_{\alpha,\beta}$ is a normalized 2-cocycle on G_ρ. Its cohomology class $\kappa_\rho \in H^2(G_\rho; K^*)$ depends only on the equivalence class of ρ and does not depend of the choice of the lifts $\{\tilde{\alpha}\}_\alpha$.*

Proof. Observe that for any $\alpha, \beta \in G_\rho$,

$$l_{\tilde{\alpha}}\, l_{\tilde{\beta}} = \theta_{\tilde{\alpha},\tilde{\beta}}\, l_{\tilde{\alpha}\tilde{\beta}} = \theta_{\tilde{\alpha},\tilde{\beta}}\, l_{\widetilde{\alpha\beta}\, h_{\alpha,\beta}} = l_{\widetilde{\alpha\beta}}\, L_{\alpha,\beta}. \tag{3.2.c}$$

We can rewrite this as

$$L_{\alpha,\beta} = l^{-1}_{\widetilde{\alpha\beta}}\, l_{\tilde{\alpha}}\, l_{\tilde{\beta}}.$$

This formula implies that for any $\alpha, \beta, \gamma \in G_\rho$,

$$L_{\alpha\beta,\gamma}\, l^{-1}_{\tilde{\gamma}}\, L_{\alpha,\beta}\, l_{\tilde{\gamma}} = L_{\alpha,\beta\gamma}\, L_{\beta,\gamma}.$$

Applying $\bar{\rho}$ to both sides, we obtain that

$$\bar{\rho}(L_{\alpha\beta,\gamma})\, \bar{\rho}(l^{-1}_{\tilde{\gamma}}\, L_{\alpha,\beta}\, l_{\tilde{\gamma}}) = \bar{\rho}(L_{\alpha,\beta\gamma})\, \bar{\rho}(L_{\beta,\gamma}). \tag{3.2.d}$$

Since $\gamma \in G_\rho$, the matrix

$$\bar{\rho}(l^{-1}_{\tilde{\gamma}}\, L_{\alpha,\beta}\, l_{\tilde{\gamma}}) = \overline{\tilde{\gamma}\rho}(L_{\alpha,\beta})$$

is conjugate to $\bar{\rho}(L_{\alpha,\beta})$. Therefore, taking the determinant on both sides of (3.2.d) we obtain $\kappa_{\alpha\beta,\gamma}\, \kappa_{\alpha,\beta} = \kappa_{\alpha,\beta\gamma}\, \kappa_{\beta,\gamma}$. Thus, the family $\{\kappa_{\alpha,\beta}\}_{\alpha,\beta}$ is a 2-cocycle. It is normalized, since $\kappa_{1,1} = 1$.

To prove that κ_ρ does not depend of the choice of the lifts $\{\tilde{\alpha}\}_\alpha$, suppose that each $\tilde{\alpha}$ is replaced with $\tilde{\alpha}' = \tilde{\alpha}\, g_\alpha$ for some $g_\alpha \in \Gamma$. We have

$$l_{\tilde{\alpha}'} = \theta^{-1}_{\tilde{\alpha},g_\alpha}\, l_{\tilde{\alpha}}\, l_{g_\alpha} = l_{\tilde{\alpha}}\, r_\alpha,$$

where

$$r_\alpha = \theta^{-1}_{\tilde{\alpha},g_\alpha}\, l_{g_\alpha} \in B_1. \tag{3.2.e}$$

For $\alpha, \beta \in G_\rho$, we have $l_{\tilde{\alpha}'}\, l_{\tilde{\beta}'} = l_{\widetilde{\alpha\beta}'}\, L'_{\alpha,\beta}$, where

$$\begin{aligned} L'_{\alpha,\beta} &= l^{-1}_{\widetilde{\alpha\beta}'}\, l_{\tilde{\alpha}'}\, l_{\tilde{\beta}'} \\ &= r^{-1}_{\alpha\beta}\, l^{-1}_{\widetilde{\alpha\beta}}\, l_{\tilde{\alpha}}\, r_\alpha\, l_{\tilde{\beta}}\, r_\beta \\ &= r^{-1}_{\alpha\beta}\, (l^{-1}_{\widetilde{\alpha\beta}}\, l_{\tilde{\alpha}}\, l_{\tilde{\beta}})\, l^{-1}_{\tilde{\beta}}\, r_\alpha\, l_{\tilde{\beta}}\, r_\beta \\ &= r^{-1}_{\alpha\beta}\, L_{\alpha,\beta}\, l^{-1}_{\tilde{\beta}}\, r_\alpha\, l_{\tilde{\beta}}\, r_\beta \in B_1. \end{aligned} \tag{3.2.f}$$

Applying $\bar{\rho}$, we obtain

$$\bar{\rho}(L'_{\alpha,\beta}) = \bar{\rho}(r_{\alpha\beta})^{-1}\, \bar{\rho}(L_{\alpha,\beta})\, \bar{\rho}(l_{\tilde{\beta}}^{-1}\, r_\alpha\, l_{\tilde{\beta}})\, \bar{\rho}(r_\beta).$$

Taking the determinant and using the inclusion $\beta \in G_\rho$, we obtain

$$\det \bar{\rho}(L'_{\alpha,\beta}) = \det \bar{\rho}(r_\alpha)\ \det \bar{\rho}(r_\beta)\ \det \bar{\rho}(r_{\alpha\beta})^{-1}\ \det \bar{\rho}(L_{\alpha,\beta}).$$

This proves that the cohomology class κ_ρ of the cocycle $\{\kappa_{\alpha,\beta}\}_{\alpha,\beta}$ does not depend on the choice of the lifts $\{\tilde{\alpha}\}_\alpha$. That κ_ρ depends only on the equivalence class of ρ follows from the definitions. □

3.3 The cohomology class ζ_ρ. Assume that a θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is a *Schur θ-representation* in the sense that the only matrices $A \in \mathrm{GL}_n(K)$ such that $A\,\rho(h) = \rho(h)\,A$ for all $h \in \Gamma$, are scalar matrices. We define $\zeta_\rho \in H^2(G_\rho; K^*)$ as follows. As above, choose $\tilde{\alpha} \in q^{-1}(\alpha)$ for each $\alpha \in G_\rho$ so that $\tilde{1} = 1$. By definition of G_ρ, for $\alpha \in G_\rho$, there is a matrix $M_\alpha \in \mathrm{GL}_n(K)$ such that

$$\tilde{\alpha}\rho = M_\alpha^{-1}\,\rho\,M_\alpha. \tag{3.3.a}$$

By the assumptions on ρ, such M_α is unique up to multiplication by an element of K^*; we fix M_α for all α. For $\alpha = 1$, we take M_α to be the unit $(n \times n)$-matrix. Recall the algebras B, $B_1 \subset B$, the algebra homomorphism $\bar{\rho}\colon B_1 \to \mathrm{Mat}_n(K)$, and the elements $L_{\alpha,\beta}$ of B_1 satisfying (3.2.b) and (3.2.c). Formula (3.2.b) implies that $L_{\alpha,\beta}$ is invertible in B_1.

3.3.1 Lemma. *For any $\alpha, \beta \in G_\rho$, there is a unique $\zeta_{\alpha,\beta} \in K^*$ such that*

$$\zeta_{\alpha,\beta}\, M_\alpha\, M_\beta = M_{\alpha\beta}\, \bar{\rho}(L_{\alpha,\beta}). \tag{3.3.b}$$

The family $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is a normalized 2-cocycle on G_ρ. Its cohomology class $\zeta_\rho \in H^2(G_\rho; K^*)$ depends only on the equivalence class of ρ and does not depend on the choice of the matrices $\{M_\alpha\}_\alpha$ or the lifts $\{\tilde{\alpha}\}_\alpha$.

Proof. For $\alpha, \beta \in G_\rho$ and $h \in \Gamma$,

$$\begin{aligned} M_{\alpha\beta}^{-1}\,\rho(h)\,M_{\alpha\beta} &= (\widetilde{\alpha\beta}\rho)(h) \\ &= \bar{\rho}(l_{\widetilde{\alpha\beta}}^{-1}\, l_h\, l_{\widetilde{\alpha\beta}}) \\ &= \bar{\rho}(L_{\alpha,\beta}\, l_{\tilde{\beta}}^{-1}\, l_{\tilde{\alpha}}^{-1}\, l_h\, l_{\tilde{\alpha}}\, l_{\tilde{\beta}}\, L_{\alpha,\beta}^{-1}) \\ &= \bar{\rho}(L_{\alpha,\beta})\, M_\beta^{-1}\, M_\alpha^{-1}\, \rho(h)\, M_\alpha\, M_\beta\, \bar{\rho}(L_{\alpha,\beta})^{-1}. \end{aligned}$$

Since this holds for all $h \in \Gamma$ and ρ is a Schur θ-representation, there is a unique $\zeta_{\alpha,\beta} \in K^*$ satisfying (3.3.b).

The rest of the proof uses a certain extension $\bar\rho_+$ of the algebra homomorphism $\bar\rho\colon B_1 \to \mathrm{Mat}_n(K)$. Consider the subgroup $G'_\rho = q^{-1}(G_\rho)$ of G' and the K-submodule B_ρ of B with basis $\{l_a \mid a \in G'_\rho\}$. Clearly, B_ρ is a subalgebra of B containing B_1. Any element of G'_ρ expands uniquely in the form $\tilde\alpha g$ with $\alpha \in G_\rho$ and $g \in \Gamma$. The formula

$$\bar\rho_+(l_{\tilde\alpha g}) = \theta^{-1}_{\tilde\alpha,g}\, M_\alpha\, \rho(g) = \theta^{-1}_{\tilde\alpha,g}\, M_\alpha\, \bar\rho(l_g) \tag{3.3.c}$$

defines a K-linear homomorphism $\bar\rho_+\colon B_\rho \to \mathrm{Mat}_n(K)$. We have $\bar\rho_+|_{B_1} = \bar\rho$ because $\bar\rho_+(l_g) = \theta^{-1}_{1,g}\, M_1\, \bar\rho(l_g) = \bar\rho(l_g)$ for any $g \in \Gamma$. By definition, $\bar\rho_+(l_{\tilde\alpha}) = M_\alpha$ for all $\alpha \in G_\rho$. Also

$$\bar\rho_+(l)\,\bar\rho_+(l') = \bar\rho_+(l\,l') \tag{3.3.d}$$

for all $l \in B_\rho$ and $l' \in B_1$. Indeed, it suffices to verify this for $l = l_{\tilde\alpha g}$ and $l' = l_h$, where $\alpha \in G_\rho$ and $g, h \in \Gamma$. We have

$$\begin{aligned}
\bar\rho_+(l_{\tilde\alpha g})\,\bar\rho_+(l_h) &= \theta^{-1}_{\tilde\alpha,g}\, M_\alpha\, \rho(g)\,\rho(h)\\
&= \theta^{-1}_{\tilde\alpha,g}\,\theta_{g,h}\, M_\alpha\, \rho(gh)\\
&= \theta^{-1}_{\tilde\alpha,g}\,\theta_{g,h}\,\theta_{\tilde\alpha,gh}\,\bar\rho_+(l_{\tilde\alpha gh})\\
&= \theta_{\tilde\alpha g,h}\,\bar\rho_+(l_{\tilde\alpha gh})\\
&= \bar\rho_+(l_{\tilde\alpha g}\, l_h).
\end{aligned}$$

Though we shall not need it in this proof, it is instructive to note that

$$\bar\rho_+(l')\,\bar\rho_+(l) = \bar\rho_+(l'\,l) \tag{3.3.e}$$

for all $l \in B_\rho$ and $l' \in B_1$. Indeed, if $l = l_{\tilde\alpha g}$ and $l' = l_h$ with $\alpha \in G_\rho$, $g, h \in \Gamma$, then

$$\begin{aligned}
\bar\rho_+(l_h)\,\bar\rho_+(l_{\tilde\alpha g}) &= \theta^{-1}_{\tilde\alpha,g}\,\rho(h)\, M_\alpha\,\rho(g)\\
&= \theta^{-1}_{\tilde\alpha,g}\, M_\alpha\, \tilde\alpha\rho(h)\,\rho(g)\\
&= \theta^{-1}_{\tilde\alpha,g}\,\bar\rho_+(l_{\tilde\alpha})\,\bar\rho_+(l_{\tilde\alpha}^{-1}\, l_h\, l_{\tilde\alpha})\,\bar\rho_+(l_g)\\
&= \theta^{-1}_{\tilde\alpha,g}\,\bar\rho_+(l_{\tilde\alpha}\, l_{\tilde\alpha}^{-1}\, l_h\, l_{\tilde\alpha}\, l_g)\\
&= \bar\rho_+(l_h\, l_{\tilde\alpha g}),
\end{aligned}$$

where the fourth equality is obtained by applying (3.3.d) twice.

Observe now that for any $\alpha, \beta \in G_\rho$,

$$\begin{aligned}
\bar\rho_+(l_{\tilde\alpha}\, l_{\tilde\beta}) &= \bar\rho_+(l_{\widetilde{\alpha\beta}}\, L_{\alpha,\beta})\\
&= \bar\rho_+(l_{\widetilde{\alpha\beta}})\,\bar\rho_+(L_{\alpha,\beta})\\
&= M_{\alpha\beta}\,\bar\rho(L_{\alpha,\beta}) = \zeta_{\alpha,\beta}\, M_\alpha\, M_\beta = \zeta_{\alpha,\beta}\,\bar\rho_+(l_{\tilde\alpha})\,\bar\rho_+(l_{\tilde\beta}),
\end{aligned} \tag{3.3.f}$$

where the second equality follows from (3.3.d). We claim that more generally, for all $a, b \in G'_\rho$,

$$\bar\rho_+(l_a\, l_b) = \zeta_{q(a),q(b)}\,\bar\rho_+(l_a)\,\bar\rho_+(l_b). \tag{3.3.g}$$

To see this, expand $a = \tilde{\alpha} g$, $b = \tilde{\beta} h$ with $\alpha = q(a)$, $\beta = q(b) \in G_\rho$ and $g, h \in \Gamma$. Then

$$\begin{aligned}
\bar{\rho}_+(l_a\, l_b) &= \bar{\rho}_+(\theta_{\tilde{\alpha},g}^{-1}\, \theta_{\tilde{\beta},h}^{-1}\, l_{\tilde{\alpha}}\, l_g\, l_{\tilde{\beta}}\, l_h) \\
&= \theta_{\tilde{\alpha},g}^{-1}\, \theta_{\tilde{\beta},h}^{-1}\, \bar{\rho}_+(l_{\tilde{\alpha}}\, l_{\tilde{\beta}}\, l_{\tilde{\beta}}^{-1}\, l_g\, l_{\tilde{\beta}}\, l_h) \\
&= \theta_{\tilde{\alpha},g}^{-1}\, \theta_{\tilde{\beta},h}^{-1}\, \bar{\rho}_+(l_{\tilde{\alpha}}\, l_{\tilde{\beta}})\, \bar{\rho}_+(l_{\tilde{\beta}}^{-1}\, l_g\, l_{\tilde{\beta}})\, \bar{\rho}_+(l_h) \\
&= \theta_{\tilde{\alpha},g}^{-1}\, \theta_{\tilde{\beta},h}^{-1}\, \zeta_{\alpha,\beta}\, M_\alpha\, M_\beta\, \tilde{\beta}\rho(g)\, \rho(h) \\
&= \zeta_{\alpha,\beta}\, \theta_{\tilde{\alpha},g}^{-1}\, \theta_{\tilde{\beta},h}^{-1}\, M_\alpha\, \rho(g)\, M_\beta\, \rho(h) \\
&= \zeta_{\alpha,\beta}\, \bar{\rho}_+(l_a)\, \bar{\rho}_+(l_b),
\end{aligned}$$

where we use formulas (3.3.a), (3.3.c), (3.3.d), (3.3.f).

Formula (3.3.g), the invertibility of the matrices $\bar{\rho}_+(l_a)$ for $a \in G'_\rho$, and the associativity of multiplication in B imply that $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is a 2-cocycle on G_ρ. It is normalized, since $\zeta_{1,1} = 1$. That the cohomology class of this cocycle does not depend on the choice of $\{M_\alpha\}_\alpha$ follows directly from (3.3.b).

If $\tilde{\alpha}$ is traded for $\tilde{\alpha}' = \tilde{\alpha}\, g_\alpha$ with $g_\alpha \in \Gamma$, then $L_{\alpha,\beta}$ is traded for $L'_{\alpha,\beta}$ given by (3.2.e), (3.2.f) and M_α is traded for $M'_\alpha = M_\alpha\, \rho(g_\alpha)$. Substituting these new values of L, M in (3.3.b), we obtain a cocycle that differs from $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ by a coboundary.

When ρ is conjugated by $M \in \mathrm{GL}_n(K)$, the group G_ρ is preserved while the matrices $\{M_\alpha\}_\alpha$ are replaced by $\{M^{-1} M_\alpha M\}_\alpha$. Formula (3.3.b) shows that the cocycle $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ is then preserved. Thus, ζ_ρ depends only on the equivalence class of ρ. □

3.3.2 Lemma. *We have $\kappa_\rho = n\, \zeta_\rho$, where $n = \dim \rho$.*

This lemma is deduced from (3.3.b) by taking the determinant.

Formula (1.3.b) generalizes to the present setting: if $K = \mathbb{C}$ and $\theta_{\alpha,\beta} \in S^1$ for all $\alpha, \beta \in \Gamma$, then

$$\zeta_\rho \in H^2(G_\rho; S^1) \subset H^2(G_\rho; \mathbb{C}^*). \tag{3.3.h}$$

The proof goes as in Section 2.4. The assumptions on θ ensure that the mapping $\Gamma \to \mathbb{R}_+, h \mapsto |\det \rho(h)|$ is a group homomorphism. Since Γ is finite, $\det \rho(h) \in S^1$ for all $h \in \Gamma$. By (3.2.b), $\det \bar{\rho}(L_{\alpha,\beta}) \in S^1$ for all $\alpha, \beta \in G_\rho$. We can also assume that $\det M_\alpha = 1$ for all $\alpha \in G_\rho$. Formula (3.3.b) implies then that $\zeta_{\alpha,\beta} \in S^1$ for all $\alpha, \beta \in G_\rho$. This yields (3.3.h).

V.4 Properties of κ_ρ and ζ_ρ

We establish a few properties of the cohomology classes κ_ρ and ζ_ρ. In particular, we show that these classes are equivariant with respect to conjugation in G and depend only on the cohomology class of the given normalized 2-cocycle $\theta = \{\theta_{a,b}\}_{a,b \in G'}$.

4.1 Lemma. *For any $\gamma \in G$ and any θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$, we have $G_{\gamma\rho} = \gamma\, G_\rho\, \gamma^{-1}$. Let $\gamma_*\colon H^2(G_\rho; K^*) \to H^2(G_{\gamma\rho}; K^*)$ be the isomorphism induced by the conjugation by γ. Then $\gamma_*(\kappa_\rho) = \kappa_{\gamma\rho}$. If ρ is a Schur θ-representation, then $\gamma_*(\zeta_\rho) = \zeta_{\gamma\rho}$.*

Proof. Fix $c \in q^{-1}(\gamma) \subset G'$. In the statement of the lemma $\gamma\rho = c\rho\colon \Gamma \to \mathrm{GL}_n(K)$. This agrees with the fact that the group $G_{c\rho}$ and the cohomology classes $\kappa_{c\rho}$ and $\zeta_{c\rho}$ do not depend on the choice of c in $q^{-1}(\gamma)$.

The equality $G_{\gamma\rho} = \gamma\, G_\rho\, \gamma^{-1}$ is obvious since $G_\rho \subset G$ is the stabilizer of the equivalence class of ρ in $\mathcal{R}_\theta$ and $G_{\gamma\rho} \subset G$ is the stabilizer of the equivalence class of $\gamma\rho = c\rho$ in $\mathcal{R}_\theta$.

For $a \in G'$, set

$$|a|_c = \frac{\theta_{c^{-1},ac}\, \theta_{a,c}}{\theta_{c,c^{-1}}} \in K^*.$$

A similar expression appears in the definition of the θ-representation $c\rho$ of Γ. By (3.1.a), for all $h \in \Gamma$,

$$c\rho(h) = |h|_c\, \rho(c^{-1}hc). \tag{4.1.a}$$

We need the following two formulas: for any $a \in G'$,

$$|a|_c\ |c^{-1}ac|_{c^{-1}} = 1 \tag{4.1.b}$$

and for any $a, b \in G'$,

$$\theta_{c^{-1}ac,\, c^{-1}bc} = |a|_c^{-1}\, |b|_c^{-1}\, |ab|_c\, \theta_{a,b}. \tag{4.1.c}$$

Formula (4.1.b) is deduced from the cocycle identity for θ. Namely,

$$\begin{aligned}
|a|_c\ |c^{-1}ac|_{c^{-1}} &= \frac{\theta_{c^{-1},ac}\, \theta_{a,c}}{\theta_{c,c^{-1}}}\ \frac{\theta_{c,c^{-1}a}\, \theta_{c^{-1}ac,c^{-1}}}{\theta_{c^{-1},c}} \\
&= \frac{\theta_{ac,c^{-1}}\, \theta_{a,c}\, \theta_{c,c^{-1}a}\, \theta_{c^{-1},a}}{\theta^2_{c,c^{-1}}} \\
&= \frac{\theta_{a,1}\, \theta_{c,c^{-1}}\, \theta_{1,a}\, \theta_{c,c^{-1}}}{\theta^2_{c,c^{-1}}} = 1,
\end{aligned}$$

where in the second equality we use that $\theta_{c^{-1},ac}\, \theta_{c^{-1}ac,c^{-1}} = \theta_{ac,c^{-1}}\, \theta_{c^{-1},a}$ and $\theta_{c,c^{-1}} = \theta_{c^{-1},c}$; in the third equality we use that $\theta_{ac,c^{-1}}\, \theta_{a,c} = \theta_{a,1}\, \theta_{c,c^{-1}}$ and $\theta_{c,c^{-1}a}\, \theta_{c^{-1},a} = \theta_{1.a}\, \theta_{c,c^{-1}}$; in the last equality we use that $\theta_{a,1} = \theta_{1,a} = 1$.

The proof of (4.1.c) is similar:

$$\begin{aligned}\theta_{c^{-1}ac,c^{-1}bc} &= \frac{\theta_{ac,c^{-1}bc}\,\theta_{c^{-1},abc}}{\theta_{c^{-1},ac}}\\ &= \frac{\theta_{a,c}\,\theta_{ac,c^{-1}bc}\,\theta_{c^{-1},abc}}{\theta_{a,c}\,\theta_{c^{-1},ac}}\\ &= \frac{\theta_{a,bc}\,\theta_{c,c^{-1}bc}\,\theta_{c^{-1},abc}}{\theta_{a,c}\,\theta_{c^{-1},ac}}\\ &= \frac{\theta_{b,c}\,\theta_{a,bc}\,\theta_{c,c^{-1}bc}\,\theta_{c^{-1},abc}}{\theta_{b,c}\,\theta_{a,c}\,\theta_{c^{-1},ac}}\\ &= \frac{\theta_{a,b}\,\theta_{ab,c}\,\theta_{c,c^{-1}bc}\,\theta_{c^{-1},abc}}{\theta_{b,c}\,\theta_{a,c}\,\theta_{c^{-1},ac}}.\end{aligned}$$

Expressing $\theta_{c,c^{-1}bc}$ from $\theta_{c,c^{-1}bc}\,\theta_{c^{-1},bc} = \theta_{1,bc}\,\theta_{c,c^{-1}} = \theta_{c,c^{-1}}$ and substituting in the previous formula, we obtain

$$\theta_{c^{-1}ac,c^{-1}bc} = \frac{\theta_{c,c^{-1}}}{\theta_{c^{-1},ac}\,\theta_{a,c}}\,\frac{\theta_{c,c^{-1}}}{\theta_{c^{-1},bc}\,\theta_{b,c}}\,\frac{\theta_{c^{-1},abc}\,\theta_{ab,c}}{\theta_{c,c^{-1}}}\,\theta_{a,b} = |a|_c^{-1}\,|b|_c^{-1}\,|ab|_c\,\theta_{a,b}.$$

Note one corollary of (4.1.a), (4.1.b): for all $h \in \Gamma$,

$$c\rho(chc^{-1}) = |chc^{-1}|_c\,\rho(h) = (|h|_{c^{-1}})^{-1}\,\rho(h). \tag{4.1.d}$$

In the following argument we use the superscript ρ (respectively, $c\rho$) to indicate that a cocycle or a matrix is associated with ρ (respectively, with $c\rho$). Fix a lift $\tilde\alpha \in q^{-1}(\alpha)$ for each $\alpha \in G_\rho$ so that $\tilde 1 = 1$. Consider the cocycle $\{\kappa^\rho_{\alpha,\beta}\}$ on G_ρ given by (3.2.a). Here and below α, β run over G_ρ. Clearly, $c\tilde\alpha c^{-1} \in G'$ is a lift of $\gamma\alpha\gamma^{-1}$ for all $\alpha \in G_\rho$. We shall use these lifts to compute the cocycle $\{\kappa^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}\}$ on $G_{\gamma\rho} = G_{c\rho} = \gamma G_\rho \gamma^{-1}$. We follow the definition of $\kappa^{c\rho}$ given in Section 3. Set

$$h^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} = chc^{-1},$$

where $h = h_{\alpha,\beta} = \widetilde{\alpha\beta}^{-1}\tilde\alpha\tilde\beta$. Then

$$\begin{aligned}\kappa^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} &= \big(\theta_{c\tilde\alpha c^{-1},c\tilde\beta c^{-1}}\,\theta^{-1}_{c\widetilde{\alpha\beta}c^{-1},h^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}}\big)^n \det c\rho(h^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}})\\ &= (\theta_{c\tilde\alpha c^{-1},c\tilde\beta c^{-1}}\,\theta^{-1}_{c\widetilde{\alpha\beta}c^{-1},chc^{-1}})^n \det c\rho(chc^{-1})\\ &= \left(\frac{|\tilde\alpha\tilde\beta|_{c^{-1}}\,|\widetilde{\alpha\beta}|_{c^{-1}}\,|h|_{c^{-1}}\,\theta_{\tilde\alpha,\tilde\beta}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}\,|\widetilde{\alpha\beta}h|_{c^{-1}}\,\theta_{\widetilde{\alpha\beta},h}}\right)^n \det c\rho(chc^{-1}).\end{aligned}$$

Using the equality $\tilde\alpha\tilde\beta = \widetilde{\alpha\beta}h$ and (4.1.d), we obtain

$$\begin{aligned}\kappa^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} &= \left(\frac{|\widetilde{\alpha\beta}|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\right)^n \left(\frac{\theta_{\tilde\alpha,\tilde\beta}}{\theta_{\widetilde{\alpha\beta},h}}\right)^n \det\rho(h)\\ &= \left(\frac{|\widetilde{\alpha\beta}|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\right)^n \kappa^{\rho}_{\alpha,\beta}.\end{aligned}$$

This shows that the 2-cocycles

$$(\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}) \mapsto \kappa^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} \quad\text{and}\quad (\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}) \mapsto \kappa^{\rho}_{\alpha,\beta}$$

on $G_{c\rho}$ differ by a coboundary. Hence $\kappa_{\gamma\rho} = \gamma_*(\kappa_\rho)$.

Let ρ be a Schur θ-representation. For all $\alpha \in G_\rho$, fix $M_\alpha = M^\rho_\alpha \in \mathrm{GL}_n(K)$ such that $\tilde\alpha\rho = M_\alpha^{-1}\rho M_\alpha$. We have the following equalities of θ-representations of Γ:

$$(c\tilde\alpha c^{-1})(c\rho) = c\tilde\alpha\rho = c(M_\alpha^{-1}\rho M_\alpha) = M_\alpha^{-1}(c\rho)M_\alpha.$$

Thus, to compute the cocycle $\zeta^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}$ associated with $c\rho$, we can use the matrix $M^{c\rho}_{\gamma\alpha\gamma^{-1}} = M_\alpha$ for all $\alpha \in G_\rho$.

Consider the vector $L_{\alpha,\beta} \in B_1$ associated with any $\alpha,\beta \in G_\rho$ via (3.2.b) and the vector

$$L^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} = \theta_{c\tilde\alpha c^{-1},c\tilde\beta c^{-1}}\,\theta^{-1}_{c\widetilde{\alpha\beta}c^{-1},chc^{-1}}\,l_{chc^{-1}} \in B_1,$$

where $h = h_{\alpha,\beta} = \widetilde{\alpha\beta}^{-1}\tilde\alpha\tilde\beta$. By (4.1.c),

$$L^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} = \frac{|\widetilde{\alpha\beta}|_{c^{-1}}\,|h|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\,\theta_{\tilde\alpha,\tilde\beta}\,\theta^{-1}_{\widetilde{\alpha\beta},h}\,l_{chc^{-1}}.$$

Applying $\overline{c\rho}\colon B_1 \to \mathrm{Mat}_n(K)$ and using (4.1.d), we obtain

$$\begin{aligned}\overline{c\rho}(L^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}) &= \frac{|\widetilde{\alpha\beta}|_{c^{-1}}\,|h|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\,\theta_{\tilde\alpha,\tilde\beta}\,\theta^{-1}_{\widetilde{\alpha\beta},h}\,c\rho(chc^{-1})\\ &= \frac{|\widetilde{\alpha\beta}|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\,\theta_{\tilde\alpha,\tilde\beta}\,\theta^{-1}_{\widetilde{\alpha\beta},h}\,\rho(h)\\ &= \frac{|\widetilde{\alpha\beta}|_{c^{-1}}}{|\tilde\alpha|_{c^{-1}}\,|\tilde\beta|_{c^{-1}}}\,\bar\rho(L_{\alpha,\beta}).\end{aligned}$$

Thus, $\overline{c\rho}(L^c_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}) \in \mathrm{GL}_n(K)$ is obtained from $\bar\rho(L_{\alpha,\beta}) \in \mathrm{GL}_n(K)$ via multiplication by a scalar coboundary. Then it follows from the definitions that the 2-cocycles

$$(\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}) \mapsto \zeta^{c\rho}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} \quad\text{and}\quad (\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}) \mapsto \zeta^{\rho}_{\alpha,\beta}$$

on $G_{c\rho}$ differ by a coboundary. Hence, $\gamma_*(\zeta_\rho) = \zeta_{\gamma\rho}$. □

4.2 Relation to θ. We now relate κ_ρ and ζ_ρ to the cohomology class of θ.

4.2.1 Lemma. *Let $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ be a θ-representation and $G'_\rho = q^{-1}(G_\rho)$.*

(a) *We have $q^*(\kappa_\rho) = n[\theta]$, where $q^*\colon H^2(G_\rho; K^*) \to H^2(G'_\rho; K^*)$ is the homomorphism induced by q and $[\theta] \in H^2(G'_\rho; K^*)$ is the cohomology class of the restriction of θ to G'_ρ.*

(b) *If ρ is a Schur θ-representation, then there is a function $D\colon G'_\rho \to \mathrm{GL}_n(K)$ such that for all $a, b \in G'_\rho$,*

$$\zeta_{q(a),q(b)}\, E_n = \theta_{a,b}\, D(ab)\, D(a)^{-1}\, D(b)^{-1}, \tag{4.2.a}$$

where E_n is the unit $n \times n$ matrix over K.

Proof. Fix the lifts $\tilde\alpha \in G'_\rho$ of all $\alpha \in G_\rho$ so that $\tilde 1 = 1$. For each $c \in G'_\rho$, we can expand $\widetilde{q(c)} = c\gamma_c$ with $\gamma_c \in \Gamma$. Pick $a, b \in G'_\rho$ and set $\alpha = q(a), \beta = q(b) \in G_\rho$. Then $\tilde\alpha = a\gamma_a$ and $l_{\tilde\alpha} = \theta^{-1}_{a,\gamma_a} l_a l_{\gamma_a}$. Substituting this expansion of $l_{\tilde\alpha}$ and similar expansions of $l_{\tilde\beta}$ and $l_{\widetilde{\alpha\beta}}$ in $L_{\alpha,\beta} = l^{-1}_{\widetilde{\alpha\beta}}\, l_{\tilde\alpha}\, l_{\tilde\beta} \in B_1$, we obtain

$$L_{\alpha,\beta} = d(a,b)\, l^{-1}_{\gamma_{ab}}\, l^{-1}_{ab}\, l_a\, l_{\gamma_a}\, l_b\, l_{\gamma_b},$$

where

$$d(a,b) = \theta_{ab,\gamma_{ab}}\, \theta^{-1}_{a,\gamma_a}\, \theta^{-1}_{b,\gamma_b}$$

is a coboundary. The equalities $l_a\, l_b = \theta_{a,b}\, l_{ab}$ and $\tilde\beta = b\gamma_b$ imply that

$$\begin{aligned}
(d(a,b))^{-1}\, L_{\alpha,\beta} &= l^{-1}_{\gamma_{ab}}\, l^{-1}_{ab}\, l_a\, l_b\, l_b^{-1}\, l_{\gamma_a}\, l_b\, l_{\gamma_b}\\
&= \theta_{a,b}\, l^{-1}_{\gamma_{ab}}\, l_b^{-1}\, l_{\gamma_a}\, l_b\, l_{\gamma_b}\\
&= \theta_{a,b}\, l^{-1}_{\gamma_{ab}}\, l_{\gamma_b}\, (l^{-1}_{\gamma_b}\, l_b^{-1})\, l_{\gamma_a}\, (l_b\, l_{\gamma_b})\\
&= \theta_{a,b}\, l^{-1}_{\gamma_{ab}}\, l_{\gamma_b}\, l^{-1}_{\tilde\beta}\, l_{\gamma_a}\, l_{\tilde\beta}.
\end{aligned}$$

Then

$$\bar\rho(L_{\alpha,\beta}) = d(a,b)\, \theta_{a,b}\, \bar\rho(l^{-1}_{\gamma_{ab}})\, \bar\rho(l_{\gamma_b})\, \bar\rho(l^{-1}_{\tilde\beta}\, l_{\gamma_a}\, l_{\tilde\beta}). \tag{4.2.b}$$

Taking the determinant on both sides, we conclude that

$$\kappa_{q(a),q(b)} = \kappa_{\alpha,\beta} = (d(a,b))^n\, \theta^n_{a,b}\, \det\bar\rho(l^{-1}_{\gamma_{ab}})\, \det\bar\rho(l_{\gamma_a})\, \det\bar\rho(l_{\gamma_b}).$$

Hence, the cocycles $\{\kappa_{q(a),q(b)}\}$ and $\{\theta^n_{a,b}\}$ on G'_ρ differ by a coboundary. This yields the claim (a) of the lemma.

If ρ is a Schur θ-representation, then

$$\zeta_{a,b}\, E_n = M_{\alpha\beta}\, \bar\rho(L_{\alpha,\beta})\, M_\beta^{-1}\, M_\alpha^{-1}.$$

Substituting here the expansion (4.2.b) of $\bar{\rho}(L_{\alpha,\beta})$ and using that $\bar{\rho}(l_{\tilde{\beta}}^{-1}\, l_{\gamma_a}\, l_{\tilde{\beta}}) = M_\beta^{-1}\, \bar{\rho}(l_{\gamma_a})\, M_\beta$, we obtain (4.2.a) with

$$D(c) = \theta_{c,\gamma_c}\, M_{q(c)}\, \bar{\rho}(l_{\gamma_c}^{-1})$$

for all $c \in G'_\rho$. □

Lemma 4.2.1 implies that $nq^*(\zeta_\rho) = n[\theta]$. This can be deduced from (a) since $\kappa_\rho = n\zeta_\rho$ or from (b) by taking the determinant. I do not know whether in general $q^*(\zeta_\rho) = [\theta]$.

4.3 Dependence on θ. We explain now that the cohomology classes κ and ζ depend only on the cohomology class of the 2-cocycle $\theta = \{\theta_{a,b}\}_{a,b\in G'}$. Pick a function $k\colon G' \to K^*$ with $k(1) = 1$ and define a 2-cocycle $\theta^{(k)} = \{\theta^{(k)}_{a,b}\}_{a,b\in G'}$ by

$$\theta^{(k)}_{a,b} = k(a)\, k(b)\, k(ab)^{-1}\, \theta_{a,b} \in K^*.$$

Every θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ determines a $\theta^{(k)}$-representation $\rho_k\colon \Gamma \to \mathrm{GL}_n(K)$ by

$$\rho_k(h) = k(h)\, \rho(h)$$

for $h \in \Gamma$. Equivalent θ-representations yield equivalent $\theta^{(k)}$-representations, so that the formula $\rho \mapsto \rho_k$ defines a mapping $\mathcal{R}_\theta \to \mathcal{R}_{\theta^{(k)}}$. It is a bijection because $(\rho_k)_{k^{-1}} = \rho$ for all ρ; here k^{-1} is the function $a \mapsto (k(a))^{-1}$ on G'.

The mapping $\rho \mapsto \rho_k$ is G'-equivariant, i.e., $a\rho_k = (a\rho)_k$ for all $a \in G'$. Indeed, for $h \in \Gamma$,

$$\begin{aligned} a\rho_k(h) &= \frac{\theta^{(k)}_{a^{-1},ha}\, \theta^{(k)}_{h,a}}{\theta^{(k)}_{a,a^{-1}}}\, \rho_k(a^{-1}ha) \\ &= \frac{k(a^{-1})\, k(ha)\, k(h)\, k(a)}{k(a^{-1}ha)\, k(ha)\, k(a)\, k(a^{-1})}\, \frac{\theta_{a^{-1},ha}\, \theta_{h,a}}{\theta_{a,a^{-1}}}\, k(a^{-1}ha)\, \rho(a^{-1}ha) \\ &= k(h)\, \frac{\theta_{a^{-1},ha}\, \theta_{h,a}}{\theta_{a,a^{-1}}}\, \rho(a^{-1}ha) \\ &= k(h)\, a\rho(h) \\ &= (a\rho)_k(h). \end{aligned}$$

The bijection $\mathcal{R}_\theta \to \mathcal{R}_{\theta^{(k)}}$, $\rho \mapsto \rho_k$ is equivariant with respect to the induced action of G. Therefore, $G_\rho = G_{\rho_k}$ for every θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$. We next verify that $\kappa_\rho = \kappa_{\rho_k}$. Fixing the lifts $\tilde{\alpha} \in G'$ of all $\alpha \in G_\rho$ as above, we represent κ_ρ

by the 2-cocycle (3.2.a) and represent κ_{ρ_k} by the 2-cocycle

$$\begin{aligned}\kappa^{(k)}_{\alpha,\beta} &= \left(\frac{\theta^{(k)}_{\tilde\alpha,\tilde\beta}}{\theta^{(k)}_{\widetilde{\alpha\beta},h_{\alpha,\beta}}}\right)^n \det(\rho_k(h_{\alpha,\beta}))\\ &= \left(\frac{k(\tilde\alpha)\,k(\tilde\beta)\,k(\tilde\alpha\tilde\beta)}{k(\tilde\alpha\tilde\beta)\,k(\widetilde{\alpha\beta})\,k(h_{\alpha,\beta})}\right)^n (k(h_{\alpha,\beta}))^n\,\kappa_{\alpha,\beta}\\ &= \left(\frac{k(\tilde\alpha)\,k(\tilde\beta)}{k(\widetilde{\alpha\beta})}\right)^n \kappa_{\alpha,\beta}.\end{aligned}$$

Thus, the cocycles $\{\kappa_{\alpha,\beta}\}$ and $\{\kappa^{(k)}_{\alpha,\beta}\}$ lie in the same cohomology class.

We claim that if ρ is a Schur θ-representation, then $\zeta_\rho = \zeta_{\rho_k}$. To see this, fix matrices $\{M_\alpha \in \mathrm{GL}_n(K)\}_\alpha$ such that $\tilde\alpha\rho = M_\alpha^{-1}\,\rho\,M_\alpha$ and consider the associated 2-cocycle $\{\zeta_{\alpha,\beta}\}$ representing ζ_ρ. It is clear that $\tilde\alpha\rho_k = (\tilde\alpha\rho)_k = M_\alpha^{-1}\,\rho_k\,M_\alpha$. A 2-cocycle $\{\zeta^{(k)}_{\alpha,\beta}\}$ representing ζ_{ρ_k} is determined from the formula

$$\zeta^{(k)}_{\alpha,\beta}\,M_\alpha\,M_\beta = M_{\alpha\beta}\,\bar\rho_k(L^{(k)}_{\alpha,\beta}),$$

where

$$L^{(k)}_{\alpha,\beta} = \theta^{(k)}_{\tilde\alpha,\tilde\beta}\,(\theta^{(k)}_{\widetilde{\alpha\beta},h_{\alpha,\beta}})^{-1}\,l_{h_{\alpha,\beta}} = \frac{k(\tilde\alpha)\,k(\tilde\beta)}{k(\widetilde{\alpha\beta})\,k(h_{\alpha,\beta})}\,L_{\alpha,\beta}.$$

Therefore

$$\bar\rho_k(L^{(k)}_{\alpha,\beta}) = k(\tilde\alpha)\,k(\tilde\beta)\,k(\widetilde{\alpha\beta})^{-1}\,\bar\rho(L_{\alpha,\beta})$$

and

$$\zeta^{(k)}_{\alpha,\beta} = k(\tilde\alpha)\,k(\tilde\beta)\,k(\widetilde{\alpha\beta})^{-1}\,\zeta_{\alpha,\beta}.$$

Hence, the cocycles $\{\zeta_{\alpha,\beta}\}$ and $\{\zeta^{(k)}_{\alpha,\beta}\}$ lie in the same cohomology class.

V.5 Equivalence of two approaches

Throughout this section we assume that the ground ring K is an algebraically closed field of characteristic zero. Fix a normalized 2-cocycle $\theta = \{\theta_{a,b} \in K^*\}_{a,b\in G'}$ on G'. The constructions above yield two different methods associating with the homomorphism $q\colon G' \to G$ an equivariant 2-cohomology class of the set of irreducible θ-representations of $\Gamma = \operatorname{Ker} q$. We summarize these methods and establish their equivalence.

5.1 Irreducible θ-representations. A θ-representation $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ is *irreducible* if the induced action of Γ on K^n preserves no submodule of K^n except 0 and K^n. It is obvious that a θ-representation equivalent to an irreducible one is itself irreducible. The set of equivalence classes of irreducible θ-representations of Γ over K will be denoted $\mathrm{Irr}_\theta(\Gamma; K)$ or, shorter, I_θ. The action of G on the set $\mathcal{R}_\theta$ of equivalence classes of θ-representations of Γ introduced in Section 3.1 preserves I_θ set-wise. This allows us to view I_θ as a G-set.

Irreducible θ-representations of Γ correspond bijectively to irreducible representations of the algebra B_1 introduced in the proof of Lemma 3.1.1. Since B_1 is a direct sum of matrix algebras over K (cf. Section IV.2), the set I_θ is finite and non-empty. By the Schur lemma, all irreducible θ-representations of Γ are Schur θ-representations in the sense of Section 3.3.

5.2 Cohomology classes $\zeta(\theta)$ and $\nabla(\theta)$. Given $i \in I_\theta$, consider the stabilizer $G_i \subset G$ of i under the action of G on I_θ. Consider the cohomology class $\zeta_i = \zeta_\rho \in H^2(G_i; K^*)$, where ρ is an arbitrary irreducible θ-representation in the equivalence class i. By Lemma 3.3.1, the class ζ_ρ does not depend on the choice of ρ. By Lemma 4.1, for any $\gamma \in G$, we have $G_{\gamma i} = \gamma G_i \gamma^{-1}$ and $\zeta_{\gamma i} = \gamma_*(\zeta_i)$, where $\gamma_*\colon H^2(G_i; K^*) \to H^2(G_{\gamma i}; K^*)$ is the isomorphism induced by the conjugation by γ. By Section II.5.2, the function $i \mapsto \zeta_i$ defines an element of the equivariant cohomology group $H^2_G(I_\theta; K^*)$. This element is denoted by $\zeta(\theta)$.

Consider the biangular G-algebra $B = \bigoplus_{\alpha\in G} B_\alpha$ derived from $q\colon G' \to G$ and θ in Example IV.2.3.4. By Lemmas IV.2.2 and IV.2.4, the G-center L of B is a semisimple crossed Frobenius G-algebra. By Section II.5, this G-algebra gives rise to a cohomology class $\nabla(\theta) \in H^2_G(I; K^*)$, where $I = \mathrm{Bas}(L)$ is the G-set of basic idempotents of L.

5.3 Lemma. *There is a canonical G-equivariant bijection $I \approx I_\theta$. Under this bijection, $\zeta(\theta) = \nabla(\theta)$.*

Proof. Let $L = \bigoplus_{\alpha\in G} L_\alpha$ be the G-center of B. As was shown in the proof of Lemma IV.2.4, the algebra $B_1 \subset B$ is a direct sum of matrix algebras over K, and $L_1 = \psi_1(B_1)$ is the center of B_1, i.e., the direct sum of the 1-dimensional centers of these matrix algebras. The basic idempotents of L are the unit elements of the matrix algebras in question. More precisely, each basic idempotent $i \in L_1$ is the unit element of a direct summand $\mathrm{Mat}_{n_i}(K)$ of B_1, where n_i is a positive integer. Let $\bar\rho_i\colon B_1 \to \mathrm{Mat}_{n_i}(K)$ be the projection onto this summand annihilating all the other direct summands of B_1. Clearly, $\bar\rho_i$ is a surjective algebra homomorphism and $\bar\rho_i(i) = E_{n_i}$, where E_n is a unit $n \times n$ matrix. Moreover, $\bar\rho_i(j) = 0$ for all basic idempotents j of L_1 distinct from i.

The homomorphism $\bar\rho_i$ determines a θ-representation $\rho_i\colon \Gamma \to \mathrm{GL}_{n_i}(K)$ by $\rho_i(h) = \bar\rho_i(l_h)$ for $h \in \Gamma$. The definition of $\bar\rho_i$ and ρ_i uses an identification of the direct summand of B_1 containing i with a matrix algebra. Any other such identification is obtained through conjugation by an invertible $n_i \times n_i$ matrix. Therefore the equivalence class of ρ_i is uniquely determined by i. We can describe ρ_i as the unique (up to equivalence) irreducible θ-representation of Γ such that $\bar{\rho}_i(i) \neq 0$.

It is clear that the mapping $I \to I_\theta$, $i \mapsto \rho_i$ is bijective. We now show that this bijection is G-equivariant. Let $\alpha \in G$ and $i \in I$. By (2.3.b) in Chapter IV,

$$\alpha i = \psi_\alpha(i) = |\Gamma|^{-1} \sum_{a \in q^{-1}(\alpha)} l_a\, i\, l_a^{-1}.$$

Observe that all summands on the right-hand side are equal to each other. Indeed, if $a \in q^{-1}(\alpha)$ and $h \in \Gamma$, then

$$l_{ah}\, i\, l_{ah}^{-1} = \theta_{a,h}\theta_{a,h}^{-1} l_a\, l_h\, i\, l_h^{-1}\, l_a^{-1} = l_a\, i\, l_a^{-1}.$$

The latter equality follows from the fact that i lies in the center of L_1 and $l_h \in L_1$. Therefore $\alpha i = l_a\, i\, l_a^{-1}$ for any $a \in q^{-1}(\alpha)$. By Section 3.1, $\overline{a\rho_i}(l) = \bar{\rho}_i(l_a^{-1}\, l\, l_a)$ for all $l \in B_1$. Setting $l = \alpha i = l_a\, i\, l_a^{-1}$, we obtain

$$\overline{a\rho_i}(\alpha i) = \bar{\rho}_i(l_a^{-1}\, l_a\, i\, l_a^{-1}\, l_a) = \bar{\rho}_i(i) \neq 0.$$

Therefore $a\rho_i = \rho_{\alpha i}$ for all $a \in q^{-1}(\alpha)$. This shows that the bijection $I \to I_\theta, i \mapsto \rho_i$ is G-equivariant. From now on, we identify I with I_θ along this bijection.

We can view $\nabla(\theta) \in H_G^2(I; K^*)$ as a function assigning to every $i \in I$ a cohomology class $\nabla_i \in H^2(G_i; K^*)$. To prove the equality $\zeta(\theta) = \nabla(\theta)$, it is enough to show that $\nabla_i = \zeta_i$ for all i. Fix $i \in I = I_\theta$. Set $n = n_i$ and $\rho = \rho_i : \Gamma \to \mathrm{GL}_n(K)$. The cohomology class ∇_i is represented by a K^*-valued 2-cocycle $\{\nabla_{\alpha,\beta}\}_{\alpha,\beta}$ on $G_i = G_\rho$ defined from the equality $s_\alpha s_\beta = \nabla_{\alpha,\beta}\, s_{\alpha\beta}$ for some non-zero vectors $s_\alpha \in i\, L_\alpha \cong K$ (cf. Section II.5.4). The cohomology class ζ_i is represented by a K^*-valued 2-cocycle $\{\zeta_{\alpha,\beta}\}_{\alpha,\beta}$ determined by a family of lifts $\{\tilde{\alpha} \in q^{-1}(\alpha)\}_{\alpha \in G_\rho}$ and matrices $\{M_\alpha\}_{\alpha \in G_\rho}$ as in Section 3.3. We fix the lifts $\tilde{\alpha}$ and the matrices M_α and verify that $\nabla_{\alpha,\beta} = \zeta_{\alpha,\beta}$ for an appropriate choice of $\{s_\alpha\}_\alpha$.

Recall the subalgebra B_ρ of B and the K-linear homomorphism $\bar{\rho}_+ : B_\rho \to \mathrm{Mat}_n(K)$ defined in the proof of Lemma 3.3.1. As we know, $\bar{\rho}_+|_{B_1} = \bar{\rho}$ so that $\bar{\rho}_+(i) = \bar{\rho}(i) = E_n$. Since $\bar{\rho}(B_1) = \mathrm{Mat}_n(K)$, for each $\alpha \in G_\rho$, there is $d_\alpha \in B_1$ such that $\bar{\rho}(d_\alpha) = M_\alpha^{-1}$. Then

$$\bar{\rho}_+(d_\alpha)\, \bar{\rho}_+(l_{\tilde{\alpha}}) = \bar{\rho}(d_\alpha)\, \bar{\rho}_+(l_{\tilde{\alpha}}) = M_\alpha^{-1}\, M_\alpha = E_n. \tag{5.3.a}$$

Set

$$s_\alpha = i\, \psi_1(d_\alpha\, l_{\tilde{\alpha}}) = |\Gamma|^{-1} \sum_{a \in \Gamma} i\, l_a\, d_\alpha\, l_{\tilde{\alpha}}\, l_a^{-1} \in i L_\alpha \subset B_\alpha.$$

By (3.3.d), (3.3.e), and (5.3.a),

$$\begin{aligned}\bar\rho_+(s_\alpha) &= |\Gamma|^{-1} \sum_{a\in\Gamma} \bar\rho_+(i\, l_a\, d_\alpha\, l_{\tilde\alpha}\, l_a^{-1}) \\ &= |\Gamma|^{-1} \sum_{a\in\Gamma} \bar\rho_+(i\, l_a)\, \bar\rho_+(d_\alpha)\, \bar\rho_+(l_{\tilde\alpha})\, \bar\rho_+(l_a^{-1}) \\ &= |\Gamma|^{-1} \sum_{a\in\Gamma} \bar\rho_+(i\, l_a)\, \bar\rho_+(l_a^{-1}) \\ &= |\Gamma|^{-1} \sum_{a\in\Gamma} \bar\rho_+(i\, l_a l_a^{-1}) \\ &= |\Gamma|^{-1} \sum_{a\in\Gamma} \bar\rho_+(i) = E_n.\end{aligned}$$

In particular, $s_\alpha \neq 0$.

Applying $\bar\rho_+$ to both sides of the equality $\nabla_{\alpha,\beta}\, s_{\alpha\beta} = s_\alpha s_\beta$ and using (3.3.g), we obtain

$$\nabla_{\alpha,\beta}\, E_n = \nabla_{\alpha,\beta}\, \bar\rho_+(s_{\alpha\beta}) = \bar\rho_+(s_\alpha s_\beta) = \zeta_{\alpha,\beta}\, \bar\rho_+(s_\alpha)\, \bar\rho_+(s_\beta) = \zeta_{\alpha,\beta}\, E_n.$$

Hence, $\nabla_{\alpha,\beta} = \zeta_{\alpha,\beta}$ for all $\alpha, \beta \in G_\rho$. □

5.4 Remark. For $\theta = 1$, the element $i \in I = \mathrm{Bas}(L)$ corresponding to an irreducible linear representation ρ of Γ can be explicitly computed by the formula

$$i = |\Gamma|^{-1} \dim\rho \sum_{h\in\Gamma} \chi_\rho(h)\, l_{h^{-1}},$$

where $\chi_\rho\colon \Gamma \to K$ is the trace of ρ; cf. [Co], Chapter 2, Theorem 5. It would be instructive to have a similar formula for all θ.

V.6 A generalization and a proof of Theorem 1.2.1

6.1 Theorem. *Let W be a closed connected oriented surface with fundamental group π. Let K be an algebraically closed field of characteristic zero. Let $\theta = \{\theta_{a,b} \in K^*\}_{a,b\in G'}$ be a normalized 2-cocycle on G' representing a cohomology class $[\theta] \in H^2(G'; K^*)$. For any group homomorphism $g\colon \pi \to G$,*

$$\sum_{g'\in \mathrm{Hom}_g(\pi,G')} (g')^*([\theta])([W]) = |\Gamma| \sum_{\substack{\rho\in \mathrm{Irr}_\theta(\Gamma;K)\\ G_\rho\supset g(\pi)}} (|\Gamma|/\dim\rho)^{-\chi(W)}\, g^*(\zeta_\rho)([W]). \tag{6.1.a}$$

Here $(g')^*([\theta])([W]) \in K^*$ is the evaluation of $(g')^*([\theta]) \in H^2(\pi; K^*)$ on $[W] \in H_2(\pi;\mathbb{Z})$ and $g^*(\zeta_\rho)([W]) \in K^*$ is the evaluation of $g^*(\zeta_\rho) \in H^2(\pi; K^*)$ on $[W]$.

For $\theta = 1$, the left-hand side of (6.1.a) is equal to $|\operatorname{Hom}_g(\pi, G')|$, and we obtain Theorem 1.2.1. For $g = 1$, Theorem 6.1 was first obtained in [Tu5].

If $g\colon \pi \to G$ is an epimorphism, then (6.1.a) may be rewritten as follows:

$$\sum_{g' \in \operatorname{Hom}_g(\pi, G')} (g')^*([\theta])([W]) = |\Gamma| \sum_{\substack{\rho \in \operatorname{Irr}_\theta(\Gamma; K) \\ G_\rho = G}} (|\Gamma| / \dim \rho)^{-\chi(W)}\, g^*(\zeta_\rho)([W]). \tag{6.1.b}$$

6.2 Proof of Theorem 6.1. Replacing, if necessary, G by $g(\pi)$ and G' by $q^{-1}(g(\pi))$, we reduce ourselves to the case where $g(\pi) = G$. It is enough therefore to prove (6.1.b).

The proof goes by computing both sides of (6.1.b) as state sums on a triangulation of W. Pick a base point $w \in W$ so that $= \pi_1(W, w)$. Fix a triangulation T of W such that w is among the vertices of T. Let $T_\bullet$ be the one-point set $\{w\}$. By Section IV.3.1, every G-system on T determines a homotopy class of mappings $W \to K(G, 1)$ carrying w to the base point of $K(G, 1)$. These mappings induce a homomorphism $\pi = \pi_1(W, w) \to G$, and any homomorphism $\pi \to G$ may be presented in this way by a G-system on T. Fix once and for all a G-system $\{g_e\}_e$ on T presenting g. Here e runs over the set $\operatorname{Edg}(T)$ of oriented edges of T.

A *labeling* of T is a mapping $\ell\colon \operatorname{Edg}(T) \to G'$ such that $\ell(e^{-1}) = (\ell(e))^{-1}$ and $q(\ell(e)) = g_e$ for all $e \in \operatorname{Edg}(T)$. A labeling ℓ is *admissible* if $\ell(e_1)\,\ell(e_2)\,\ell(e_3) = 1$ for any three consecutive oriented edges e_1, e_2, e_3 forming the boundary of a face (a 2-simplex) of T. In other words, a labeling is admissible if it is a G'-system on T. Denote the set of labelings of T by $\mathcal{L}$ and denote the subset of $\mathcal{L}$ formed by the admissible labelings by $\mathcal{L}_0$.

Given a labeling $\ell \in \mathcal{L}$, we assign to any path p in T formed by consecutive oriented edges $e_1, \dots, e_N$ the product $\ell(p) = \ell(e_1)\,\ell(e_2) \dots \ell(e_N) \in G'$. For admissible ℓ, this product is a homotopy invariant of p: if two paths p, p' have the same endpoints and are homotopic relative to the endpoints, then $\ell(p) = \ell(p')$. Applying the mapping $p \mapsto \ell(p)$ to loops based at w, we obtain a homomorphism $\pi \to G'$ denoted by $\phi(\ell)$. It is clear that $q \circ \phi(\ell) = g$ so that $\phi(\ell) \in \operatorname{Hom}_g(\pi, G')$. The formula $\ell \mapsto \phi(\ell)$ defines a mapping $\phi\colon \mathcal{L}_0 \to \operatorname{Hom}_g(\pi, G')$.

Denote by k_0, k_1, k_2 the number of vertices, edges, and faces of T, respectively. We claim that for any $g' \in \operatorname{Hom}_g(\pi, G')$,

$$|\phi^{-1}(g')| = |\Gamma|^{k_0 - 1}. \tag{6.2.a}$$

To see this, pick an arbitrary spanning tree $A \subset W$ formed by all vertices and $k_0 - 1$ edges of T; here we use that W is connected. For every vertex v of T, there is a path p_v in A formed by oriented edges of A and leading from w to v. Since A is a tree, the path p is unique up to homotopy. Any oriented edge e of T not lying in A determines a loop

$p_{i_e} e\, p_{t_e}^{-1}$, where i_e and t_e are the initial and the terminal endpoints of e, respectively. The homotopy classes of such loops corresponding to all oriented edges e of T not lying in A generate $\pi = \pi_1(W, w)$. Therefore the set $\phi^{-1}(g')$ consists of the labelings $\ell \in \mathcal{L}_0$ such that for all e as above,

$$\ell(p_{i_e} e\, p_{t_e}^{-1}) = g'(p_{i_e} e\, p_{t_e}^{-1}).$$

This equality may be rewritten as

$$\ell(e) = (\ell(p_{i_e}))^{-1}\; g'(p_{i_e} e\, p_{t_e}^{-1})\; \ell(p_{t_e}). \tag{6.2.b}$$

Thus, to specify $\ell \in \phi^{-1}(g')$, we can assign arbitrary labels to the $k_0 - 1$ edges of A oriented away from w, the inverse labels to the same edges oriented towards w, and use (6.2.b) to label the oriented edges of T not lying in A. The resulting labeling of T is necessarily admissible. This implies (6.2.a).

For every admissible labeling ℓ of T, we compute $(\phi(\ell))^*([\theta])([W])$ as follows. Fix a total order $<$ on the set of vertices of T. Any face Δ of T has three vertices P, Q, R with $P < Q < R$. Set $\varepsilon_\Delta(W) = +1$ if the orientation of Δ (induced by that of W) induces the direction from P to Q on the edge $PQ \subset \partial\Delta$ and set $\varepsilon_\Delta(W) = -1$ otherwise. Let $\ell_1^\Delta = \ell(PQ) \in G'$ and $\ell_2^\Delta = \ell(QR) \in G'$ be the labels of the edges PQ, QR oriented from P to Q and from Q to R, respectively. Then

$$(\phi(\ell))^*([\theta])([W]) = \prod_\Delta (\theta_{\ell_1^\Delta, \ell_2^\Delta})^{\varepsilon_\Delta(W)} \in K^*,$$

where Δ runs over all faces of T.

Formula (6.2.a) and the results of the previous paragraph allow us to compute the left-hand side of (6.1.b) as follows:

$$\begin{aligned} \sum_{g' \in \mathrm{Hom}_g(\pi, G')} (g')^*([\theta])([W]) &= |\Gamma|^{1-k_0} \sum_{\ell \in \mathcal{L}_0} (\phi(\ell))^*([\theta])([W]) \\ &= |\Gamma|^{1-k_0} \sum_{\ell \in \mathcal{L}_0} \prod_\Delta (\theta_{\ell_1^\Delta, \ell_2^\Delta})^{\varepsilon_\Delta(W)}. \end{aligned} \tag{6.2.c}$$

We now compute the right-hand side of (6.1.b). Let $B = \bigoplus_{\alpha \in G} B_\alpha$ be the biangular G-algebra derived from $q\colon G' \to G$ and θ in Example IV.2.3.4. Let L be the G-center of B. Lemma 5.3 and the discussion at the end of Section IV.2 compute the basic triple of L to be (I, ζ, F), where $I = \mathrm{Irr}_\theta(\Gamma; K)$ is the G-set of equivalence classes of irreducible θ-representations of Γ over K,

$$\zeta = \zeta(\theta) \in H_G^2(I; K^*)$$

is the cohomology class defined in Section 5.2, and $F(\rho) = (\dim \rho)^2$ for all $\rho \in I$.

Formula (3.5.a) in Chapter III implies that

$$\tau_B(-W) = \sum_{\substack{\rho \in I \\ G_\rho = G}} (\dim \rho)^{\chi(W)}\, g^*(\zeta_\rho)([W]).$$

Therefore the right-hand side of (6.1.b) is equal to $|\Gamma|^{1-\chi(W)}\,\tau_B(-W)$. Comparing with (6.2.c), we conclude that to finish the proof it is enough to show that

$$\tau_B(-W) = |\Gamma|^{\chi(W)-k_0} \sum_{\ell\in\mathcal{L}_0} \prod_{\Delta} (\theta_{\ell_1^\Delta,\ell_2^\Delta})^{\varepsilon_\Delta(W)}.$$

Since $\varepsilon_\Delta(W) = -\varepsilon_\Delta(-W)$, the latter formula is equivalent to the following one:

$$\tau_B(W) = |\Gamma|^{\chi(W)-k_0} \sum_{\ell\in\mathcal{L}_0} \prod_{\Delta} (\theta_{\ell_1^\Delta,\ell_2^\Delta})^{-\varepsilon_\Delta(W)}. \tag{6.2.d}$$

We now prove (6.2.d). To this end, we associate with each labeling $\ell\in\mathcal{L}$ an element $\langle\ell\rangle$ of K^* as follows. By Section II.2.3.2, we have $\theta_{\alpha,\alpha^{-1}} = \theta_{\alpha^{-1},\alpha}$ for all $\alpha\in G$. Therefore for any oriented edge e of T, the expression $\theta_{\ell(e),\ell(e^{-1})} = \theta_{\ell(e^{-1}),\ell(e)} \in K^*$ does not depend on the orientation of e and may be associated with the underlying unoriented edge. Set

$$\langle\ell\rangle = \prod_e \theta_{\ell(e),\ell(e^{-1})} \in K^*,$$

where e runs over all non-oriented edges of T.

Recall that a flag of T is a pair (a face Δ of T, a side e of Δ). Let $\{B_f\}_f$ be a set of copies of B labeled by the flags $f=(\Delta,e)$ of T. With a labeling $\ell\in\mathcal{L}$ we associate a vector $V(\ell)\in\otimes_f B_f$ as follows. For a flag $f=(\Delta,e)$, orient e so that Δ lies on its right. Let $\ell(f)\in G'$ be the value of ℓ on thus oriented edge e and let $l_{\ell(f)}$ be the corresponding basis vector of $B_f=B$. Set $V(\ell)=\otimes_f l_{\ell(f)}\in\otimes_f B_f$.

Recall the vector $\eta_\alpha^-\in B_\alpha\otimes B_{\alpha^{-1}}$ defined for all $\alpha\in G$ in Section IV.1.2 and the vector $\eta_g^- = \otimes_e \eta_{g_e}^- \in \otimes_f B_f$ defined in Section IV.3.2. Formula (2.3.a) in Chapter IV computes η_α^- and implies that

$$\eta_g^- = |\Gamma|^{-k_1} \sum_{\ell\in\mathcal{L}} \langle\ell\rangle^{-1}\, V(\ell).$$

By definition, $\tau_B(W) = D_g(\eta_g^-)$, where $D_g\colon \otimes_f B_f \to K$ is the homomorphism defined in Section IV.3.2. We now analyze D_g in more detail.

We say that a trilinear form $\Lambda\colon B\otimes B\otimes B\to K$ is *cyclically symmetric* if $\Lambda(a_1\otimes a_2\otimes a_3) = \Lambda(a_3\otimes a_1\otimes a_2)$ for all $a_1,a_2,a_3\in B$. Such a form Λ induces a homomorphism $\widetilde\Lambda\colon \bigotimes_f B_f\to K$ as follows. Every face Δ of M has three consecutive edges e_1,e_2,e_3 oriented so that Δ lies on their right. Since $B_{(\Delta,e_i)}=B$ for $i=1,2,3$, the form Λ induces a trilinear form

$$\Lambda_\Delta\colon B_{(\Delta,e_1)}\otimes B_{(\Delta,e_2)}\otimes B_{(\Delta,e_3)}\to K.$$

This form is cyclically symmetric and therefore independent of the numeration of the edges of Δ. The tensor product $\widetilde\Lambda = \otimes_\Delta \Lambda_\Delta$ over the faces Δ of T is a homomorphism $\bigotimes_f B_f\to K$.

By definition, $D_g=\widetilde\Lambda$, where $\Lambda\colon B\otimes B\otimes B\to K$ is the homomorphism $a_1\otimes a_2\otimes a_3\mapsto \eta(a_1a_2a_3,1_B)$, where $a_1,a_2,a_3\in B$ and η is the canonical inner product on B.

By the results of Section IV.2.3.4, for any $a, b \in G'$, we have $\eta(l_a, l_b) = \theta_{a,b}\,|\Gamma|$ if $ab = 1$ and $\eta(l_a, l_b) = 0$ otherwise. We define a homomorphism $\lambda\colon B \otimes B \otimes B \to K$ by the formula

$$\lambda(l_a \otimes l_b \otimes l_c) = \begin{cases} 0 & \text{if } abc \neq 1, \\ \theta_{a,b}\,\theta_{ab,c} & \text{if } abc = 1, \end{cases}$$

for any $a, b, c \in G'$. It is clear that $\Lambda = |\Gamma|\,\lambda$. The cyclic symmetry of Λ implies that λ is cyclically symmetric. Then

$$\tau_B(W) = D_g(\eta_g^-) = \widetilde{\Lambda}(\eta_g^-) = |\Gamma|^{k_2}\,\tilde{\lambda}(\eta_g^-) = |\Gamma|^{k_2-k_1} \sum_{\ell \in \mathcal{L}} \langle \ell \rangle^{-1}\,\tilde{\lambda}(V(\ell)).$$

It is clear that $k_2 - k_1 = \chi(M) - k_0$ and $\tilde{\lambda}(V(\ell)) = 0$ for non-admissible ℓ. Hence

$$\tau_B(W) = |\Gamma|^{\chi(W)-k_0} \sum_{\ell \in \mathcal{L}_0} \langle \ell \rangle^{-1}\,\tilde{\lambda}(V(\ell)). \tag{6.2.e}$$

We now fix $\ell \in \mathcal{L}_0$ and compute $\tilde{\lambda}(V(\ell))$. For any face Δ of T, set

$$v_\Delta = \otimes_e\, l_{\ell(e)} \in \otimes_e\, B_{(\Delta,e)},$$

where e runs over the three sides of Δ oriented so that Δ lies on their right. Then $V(\ell) = \otimes_\Delta v_\Delta$, where Δ runs over the faces of T, and

$$\tilde{\lambda}(V(\ell)) = \prod_\Delta \lambda_\Delta(v_\Delta).$$

To compute $\lambda_\Delta(v_\Delta)$, we use the fixed total order $<$ on the set of vertices of T. Let $P < Q < R$ be the vertices of Δ. Set

$$g_1 = \ell_1^\Delta = \ell(PQ) \in G', \quad g_2 = \ell_2^\Delta = \ell(QR) \in G', \quad g_3 = \ell_3^\Delta = \ell(RP) \in G'.$$

If $\varepsilon_\Delta(W) = -1$, then Δ lies on the right of the consecutive edges PQ, QR, and RP oriented from P to Q, from Q to R, and from R to P, respectively. Then

$$v_\Delta = l_{g_1} \otimes l_{g_2} \otimes l_{g_3} \in B_{(\Delta,PQ)} \otimes B_{(\Delta,QR)} \otimes B_{(\Delta,RP)}$$

and

$$\lambda_\Delta(v_\Delta) = \theta_{g_1,g_2}\,\theta_{g_1g_2,g_3} = \theta_{g_1,g_2}\,\theta_{g_3^{-1},g_3} = \theta_{g_1,g_2}\,\theta_{g_3,g_3^{-1}},$$

where we use the equality $g_1g_2g_3 = 1$. If $\varepsilon_\Delta(W) = 1$, then Δ lies on the right of the consecutive edges PR, RQ, and QP oriented from P to R, from R to Q, and from Q to P, respectively. Then

$$v_\Delta = l_{g_3^{-1}} \otimes l_{g_2^{-1}} \otimes l_{g_1^{-1}} \in B_{(\Delta,PR)} \otimes B_{(\Delta,RQ)} \otimes B_{(\Delta,QP)}$$

and

$$\lambda_\Delta(v_\Delta) = \theta_{g_3^{-1},g_2^{-1}}\,\theta_{g_3^{-1}g_2^{-1},g_1^{-1}} = \theta_{g_1g_2,g_2^{-1}}\,\theta_{g_1,g_1^{-1}} = \theta_{g_1,g_2}^{-1}\,\theta_{g_1,g_1^{-1}}\,\theta_{g_2,g_2^{-1}}.$$

The last equality follows from the cocycle identity (1.2.a) in Chapter II applied to $\alpha = g_1, \beta = g_2, \gamma = g_2^{-1}$. In both cases, set

$$\mu_\Delta = \lambda_\Delta(v_\Delta)\, \theta_{g_1,g_2}^{\varepsilon_\Delta(W)} = \begin{cases} \theta_{g_3,g_3^{-1}} & \text{if } \varepsilon_\Delta(W) = -1, \\ \theta_{g_1,g_1^{-1}}\, \theta_{g_2,g_2^{-1}} & \text{if } \varepsilon_\Delta(W) = +1. \end{cases}$$

Then

$$\tilde{\lambda}(V(\ell)) = \prod_\Delta \lambda_\Delta(v_\Delta) = \prod_\Delta \mu_\Delta \quad \prod_\Delta (\theta_{\ell_1^\Delta,\ell_2^\Delta})^{-\varepsilon_\Delta(W)}, \tag{6.2.f}$$

where Δ runs over all faces of T.

We claim that $\prod_\Delta \mu_\Delta = \langle \ell \rangle$. Note that the product $\prod_\Delta \mu_\Delta$ expands as a product of the expressions $\theta_{\ell(e),\ell(-e)}$ associated with edges e of T. We show that every edge $e = PQ$ of T with $P < Q$ contributes exactly one such expression. Set $\alpha = \ell(PQ) \in G'$. The edge PQ is incident to two faces $\Delta = PQR$ and $\Delta' = PQR'$ of T. We choose the notation so that the orientation of W restricted to Δ and Δ' induces on PQ the direction from P to Q and from Q to P, respectively. If $Q < R$, then $\varepsilon_\Delta(W) = 1$, $\alpha = \ell_1^\Delta$, and PQ contributes the factor $\theta_{\alpha,\alpha^{-1}}$ to μ_Δ. If $R < P$, then $\varepsilon_\Delta(W) = 1$, $\alpha = \ell_2^\Delta$, and PQ contributes the factor $\theta_{\alpha,\alpha^{-1}}$ to μ_Δ. Finally, if $P < R < Q$, then $\varepsilon_\Delta(W) = -1$, $\alpha = \ell_3^\Delta$, and $\mu_\Delta = \theta_{\alpha,\alpha^{-1}}$. A similar computation shows that PQ contributes no factors to $\mu_{\Delta'}$. Therefore

$$\prod_\Delta \mu_\Delta = \prod_e \theta_{\ell(e),\ell(-e)} = \langle \ell \rangle.$$

By (6.2.f),

$$\tilde{\lambda}(V(\ell)) = \langle \ell \rangle \prod_\Delta (\theta_{\ell_1^\Delta,\ell_2^\Delta})^{-\varepsilon_\Delta(W)}.$$

Substituting this expression in (6.2.e), we obtain (6.2.d) and the claim of the theorem.

V.7 A homological obstruction to lifting

Let W be a closed connected oriented surface of positive genus. We discuss a homological obstruction to the existence of a lift of a homomorphism $g\colon \pi = \pi_1(W) \to G$ to G'. Replacing G and G' by $g(\pi)$ and $q^{-1}(g(\pi))$, respectively, we can reduce ourselves to the case where g is an epimorphism. We shall therefore consider only this case.

7.1 A homological obstruction. If an epimorphism $g\colon \pi \to G$ lifts to G', then $g_*([W]) \in H_2(G;\mathbb{Z})$ necessarily belongs to the image of the homomorphism $q_*\colon H_2(G';\mathbb{Z}) \to H_2(G;\mathbb{Z})$ induced by $q\colon G' \to G$. If the latter condition is satisfied, then we say that the homological obstruction to the lifting of g to G' is trivial.

In general, the triviality of the homological obstruction may not imply that g lifts to G'; see Section 7.6. The next two theorems show that if the group $\Gamma = \mathrm{Ker}(q\colon G' \to G)$ is abelian or the genus of W is big enough, then there are no further obstructions.

In the sequel the symbol $[\Gamma, G']$ denotes the subgroup of Γ generated by the commutators of elements of Γ with elements of G'.

7.1.1 Theorem. *Suppose that the group Γ is abelian. An epimorphism $g\colon \pi \to G$ lifts to G' if and only if the homological obstruction to the lifting is trivial. Moreover, if g has a lift to G', then the number of such lifts is equal to*

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma|^b \, |[\Gamma, G']|^{-1}, \tag{7.1.a}$$

where $b = 2 - \chi(W)$ is the first Betti number of W.

Formula (7.1.a) can be rewritten as

$$|\operatorname{Hom}_g(\pi, G')| = |\operatorname{Hom}(\pi, \Gamma)| \times |[\Gamma, G']|^{-1}. \tag{7.1.b}$$

This formula does not directly extend to groups π distinct from the fundamental groups of closed oriented surfaces. For instance, if π is a free group of rank n, then the left-hand and right-hand sides of (7.1.b) are equal respectively to $|\Gamma|^n$ and $|\Gamma|^n \, |[\Gamma, G']|^{-1}$. These numbers are equal if and only if $[\Gamma, G'] = 1$, i.e., if and only if Γ lies in the center of G'.

7.1.2 Theorem. *Suppose that the first Betti number $b = 2 - \chi(W)$ of W satisfies*

$$b > \log_2(|[\Gamma, G']| - 1). \tag{7.1.c}$$

An epimorphism $g\colon \pi = \pi_1(W) \to G$ lifts to G' if and only if the homological obstruction to the lifting is trivial. Moreover, if g lifts to G', then

$$|\operatorname{Hom}_g(\pi, G')| \geq |\Gamma|^b \, |[\Gamma, G']|^{-1} \times \left(1 - \frac{|[\Gamma, G']| - 1}{2^b}\right). \tag{7.1.d}$$

It is understood that if $[\Gamma, G'] = 1$, then condition (7.1.c) is empty.

In the next subsection we introduce certain numerical functions $v_1, v_2, \ldots$ on $H_2(G; \mathbb{Z})$ needed in the proof of Theorems 7.1.1 and 7.1.2. Then we prove these theorems and Corollary 1.3.3.

7.2 The functions $\{v_k\}_k$. We derive from the epimorphism $q\colon G' \to G$ a sequence of numerical functions $v_1, v_2, \ldots$ on $H_2(G; \mathbb{Z})$. By definition,

$$v_k(h) = \sum_{\substack{\rho \in \operatorname{Irr}(\Gamma; \mathbb{C}) \\ \dim \rho = k,\, G_\rho = G}} \zeta_\rho(h) \in \mathbb{C},$$

for all $h \in H_2(G; \mathbb{Z})$ and $k = 1, 2, \ldots$. Starting from a certain k, the functions v_k are equal to zero. Indeed, if $k > |\Gamma/Z(\Gamma)|^{1/2}$, where $Z(\Gamma)$ is the center of Γ, then

$v_k = 0$ because the dimensions of irreducible complex representations of Γ can not exceed $|\Gamma/Z(\Gamma)|^{1/2}$. Similarly, $v_k = 0$ if k does not divide $|\Gamma/Z(\Gamma)|$.

Theorem 1.2.1 implies that for any epimorphism $g\colon \pi = \pi_1(W) \to G$,

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma|^{1-\chi(W)} \sum_{k\geq 1} v_k(g_*([W]))\, k^{\chi(W)}. \tag{7.2.a}$$

The next lemma summarizes the properties of the functions $v_1, v_2, \dots$.

7.2.1 Lemma. *For $k = 1, 2, \dots$, let N_k be the number of equivalence classes of irreducible k-dimensional complex linear representations ρ of $\Gamma = \operatorname{Ker}(q\colon G' \to G)$ such that $G_\rho = G$. Let Q be the image of the homomorphism $q_*\colon H_2(G';\mathbb{Z}) \to H_2(G;\mathbb{Z})$. Then*

(a) *the functions $v_1, v_2, \dots$ take only integer values and are zero outside Q;*
(b) *$v_k(0) = N_k$ for all $k = 1, 2, \dots$;*
(c) *for all $h \in Q$ and all k, we have $|v_k(h)| \leq N_k$ and $v_k(-h) = v_k(h)$;*
(d) *for all $h, h' \in Q$, and $k = 1, 2, \dots$, we have $v_k(h + kh') = v_k(h)$;*
(e) *for all $h \in Q$, we have $v_1(h) = N_1 = |\Gamma/[\Gamma, G']| \geq 1$.*

Proof. (a) It is well known that for any $h \in H_2(G;\mathbb{Z})$ there are a (closed connected oriented) surface W and a homomorphism $g\colon \pi_1(W) \to G$ such that $g_*([W]) = h$; see, for instance, [Zi]. We say that such a pair (W, g) *realizes* h. Consider a (finite-dimensional) complex linear representation ρ of Γ with $G_\rho \neq G$. Adding to W a handle and mapping its meridian to $1 \in G$ and its longitude to any element of $G - G_\rho$, we obtain a realization of h by a pair (a surface, a homomorphism of its fundamental group to G) such that the image of the homomorphism is not contained in G_ρ. Repeating this process, we can realize h by a pair (a surface W with fundamental group π, a homomorphism $g\colon \pi \to G$) such that $g(\pi) \subset G_\rho$ if and only if $G_\rho = G$. Formula (1.3.a) implies then that

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma|^{2d-1} \sum_{k\geq 1} v_k(h)\, k^{2-2d}, \tag{7.2.b}$$

where d is the genus of W. Any surface of a bigger genus admits a degree one map to W. Such a map induces a surjection of the fundamental groups. We can apply (7.2.b) to the composition of this surjection with g. This implies that $\sum_{k\geq 1} v_k(h)\, k^{2-2n} \in \mathbb{Q}$ for all $n \geq d$. By linear algebra, $v_k(h) \in \mathbb{Q}$ for all k.

Lemma 2.3.2 implies that for any irreducible complex linear representation ρ of Γ with $G_\rho = G$, the number $\zeta_\rho(h) \in \mathbb{C}$ is a root of unity. Thus, $v_k(h)$ is a sum of roots of unity, hence, an algebraic integer. Therefore $v_k(h) \in \mathbb{Z}$ for all k.

If $h \notin Q$, then a homomorphism g realizing h as above cannot lift to G'. Formula (7.2.b) and the argument after this formula show that $\sum_{k\geq 1} v_k(h)\, k^{2-2n} = 0$ for all sufficiently big natural numbers n. This gives a non-degenerate system of linear equations satisfied by the numbers $\{v_k(h)\}_k$. Therefore $v_k(h) = 0$ for all k.

(b) The equality $v_k(0) = N_k$ follows from the equality $\zeta_\rho(0) = 1$ for all irreducible complex linear representations ρ of Γ.

(c) The proof of (a) shows that $v_k(h)$ is a sum of N_k complex roots of unity. Hence $|v_k(h)| \leq N_k$.

If ρ is a k-dimensional complex linear representation of Γ with $G_\rho = G$, then $\zeta_\rho(-h) = (\zeta_\rho(h))^{-1} = \overline{\zeta_\rho(h)}$. This and the inclusion $v_k(h) \in \mathbb{Z}$ imply that $v_k(-h) = \overline{v_k(h)} = v_k(h)$.

(d) We have $(k\zeta_\rho)(h') = 1$ since $q^*(k\zeta_\rho) = 1$ (Lemma 2.3.3). Thus,

$$\zeta_\rho(h + kh') = \zeta_\rho(h)\,\zeta_\rho(kh') = \zeta_\rho(h)\,(k\zeta_\rho)(h') = \zeta_\rho(h).$$

Therefore $v_k(h + kh') = v_k(h)$.

(e) By definition, $v_1(h) = \sum_\rho \zeta_\rho(h)$, where ρ runs over all homomorphisms $\Gamma \to \mathbb{C}^*$ such that $G_\rho = G$. Lemma 2.3.3 and the assumption $h \in Q$ imply that $\zeta_\rho(h) = 1$ for all such ρ. Therefore $v_1(h) = N_1$. The condition $G_\rho = G$ holds if and only if $\rho(aha^{-1}h^{-1}) = 1$ for all $a \in G'$ and $h \in \Gamma$. The latter holds if and only if $\rho([\Gamma, G']) = 1$. Thus, $N_1 = |\Gamma/[\Gamma, G']|$. □

7.3 Proof of Theorem 7.1.1. We need only to prove the "if" part. Assume that the homological obstruction in question is trivial so that $h = g_*([W]) \in \operatorname{Im} q_*$. By Lemma 7.2.1(e), we have $v_1(h) = |\Gamma/[\Gamma, G']| > 0$. Since Γ is abelian, all irreducible complex linear representations of Γ are one-dimensional. So, $v_k(h) = 0$ for $k \geq 2$. Now, the claim of the theorem directly follows from (7.2.a).

7.4 Proof of Theorem 7.1.2. For $k = 1, 2, \ldots$, denote by M_k the number of equivalence classes of irreducible k-dimensional complex linear representations of Γ. It is clear that $\sum_{k\geq 1} M_k k^2 = |\Gamma|$.

Pick any $h \in H_2(G; \mathbb{Z})$. Let N_k be the same number as in Lemma 7.2.1. The inequality $|v_k(h)| \leq N_k$ established in Lemma 7.2.1 and the obvious inequality $N_k \leq M_k$ imply that for any integer $n < 2$,

$$\begin{aligned}
\sum_{k\geq 1} v_k(h)\,k^n &\geq v_1(h) - \sum_{k\geq 2} |v_k(h)|\,k^n \\
&\geq v_1(h) - \sum_{k\geq 2} M_k\,k^n \\
&= v_1(h) - \sum_{k\geq 2} M_k k^2\,k^{n-2} \\
&\geq v_1(h) - \Big(\sum_{k\geq 2} M_k k^2\Big)2^{n-2} \\
&= v_1(h) - 2^{n-2}\,(|\Gamma| - M_1) \\
&\geq v_1(h) - 2^{n-2}\,(|\Gamma| - N_1).
\end{aligned}$$

If $h \in \operatorname{Im} q_*$, then $v_1(h) = N_1 = |\Gamma|/|[\Gamma, G']|$ by Lemma 7.2.1. This gives

$$\sum_{k\geq 1} v_k(h)\, k^n \geq |\Gamma/[\Gamma, G']|\big(1 - \tfrac{|[\Gamma,G']|-1}{2^{2-n}}\big).$$

Setting here $n = \chi(W) = 2 - b$ and combining with (7.2.a), we obtain the following: if $h = g_*([W]) \in \operatorname{Im} q_*$, then

$$|\operatorname{Hom}_g(\pi, G')| \geq |\Gamma|^b\, |[\Gamma, G']|^{-1} \times \big(1 - \tfrac{|[\Gamma,G']|-1}{2^b}\big).$$

Under the assumption (7.1.c), the right-hand side is positive. This proves the first claim of the theorem. The second claim is now obvious.

7.5 Proof of Corollary 1.3.3. Replacing G and G' by $g(\pi)$ and $q^{-1}(g(\pi))$, respectively, we can reduce ourselves to the case where g is an epimorphism. Let us rewrite (7.2.a) in the following form:

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma| \sum_{k\geq 1} v_k(g_*([W]))\,\big(\tfrac{|\Gamma|}{k}\big)^{2d-2}.$$

Here $v_k(g_*([W])) \in \mathbb{Z}$ by Lemma 7.2.1(a). Also, $v_k = 0$ if k does not divide $|\Gamma/Z(\Gamma)|$. Therefore the number $|\operatorname{Hom}_g(\pi, G')|$ is divisible by $|\Gamma|\,|Z(\Gamma)|^{2d-2}$.

7.6 Example. We give an example showing that, in general, the homological obstruction introduced in Section 7.1 is insufficient to detect the existence of lifts of a homomorphism $\pi_1(W) \to G$ to G'. Let $n \geq 2$ and Γ be the finite nilpotent group with $2n$ generators $x_1, \ldots, x_{2n}$ subject to the relations $x_i^2 = 1$ and $[[x_i, x_j], x_k] = 1$ for all i, j, k (here $[x, y] = xyx^{-1}y^{-1}$). Let F be the free group with $2d \geq 2$ generators $\alpha_1, \beta_1, \ldots, \alpha_d, \beta_d$. Set

$$\gamma = \prod_{i=1}^{n} [x_{2i-1}, x_{2i}] \in [\Gamma, \Gamma] \quad \text{and} \quad \delta = \prod_{j=1}^{d} [\alpha_j, \beta_j] \in F.$$

Consider the direct product $\Gamma \times F$ and its smallest normal subgroup $\langle(\gamma, \delta)\rangle$ containing (γ, δ). Set

$$G' = (\Gamma \times F)/\langle(\gamma, \delta)\rangle,$$

and denote the projection $\Gamma \times F \to G'$ by φ. Clearly, $\varphi(\Gamma)$ is a normal subgroup of G'. Set $G = G'/\varphi(\Gamma) = F/\langle\delta\rangle$, and denote the projection $G' \to G$ by q. It is clear that G is isomorphic to the fundamental group π of a closed connected oriented surface of genus d. An isomorphism $\pi \cong G$ lifts to a homomorphism $\pi \to G'$ if and only if q has a section. We claim the following.

(i) The homomorphism $\varphi|_\Gamma\colon \Gamma \to G'$ is injective and $\operatorname{Ker} q = \varphi(\Gamma) \cong \Gamma$.

(ii) The projection $q\colon G' \to G$ induces an epimorphism $H_2(G';\mathbb{Z}) \to H_2(G;\mathbb{Z})$ so that the homological obstruction to the existence of a section of q is trivial.

(iii) If $d < n$, then q has no sections.

We outline a proof of these claims.

(i) Any $a \in \langle(\gamma, \delta)\rangle$ is a product of several conjugates of $(\gamma, \delta)^{\pm 1}$. Since γ lies in the center of Γ, we have $a = \gamma^r \prod_s f_s \delta^{\varepsilon_s} f_s^{-1}$, where s runs over a finite set of indices, $f_s \in F$, $\varepsilon_s = \pm 1$, and $r = \sum_s \varepsilon_s$. If $a \in \Gamma$, then $\prod_s f_s \delta^{\varepsilon_s} f_s^{-1} = 1$ in F. Interpreting F as the fundamental group of a punctured surface of genus d and using the maximal abelian covering, one easily deduces from the latter equality that $r = \sum_s \varepsilon_s = 0$. Then $a = \gamma^r = 1$. Hence, $\varphi|_\Gamma$ is an injection.

(ii) Consider the exact sequence

$$H_2(G'; \mathbb{Z}) \to H_2(G; \mathbb{Z}) \to \Gamma/[\Gamma, G'] \to H_1(G'; \mathbb{Z}) \to H_1(G; \mathbb{Z}).$$

To prove (ii), it suffices to show that the homomorphism $\Gamma/[\Gamma, G'] \to H_1(G'; \mathbb{Z})$ in this sequence is injective. This follows from the obvious equalities $[\Gamma, G'] = [\Gamma, \Gamma]$ and $H_1(G'; \mathbb{Z}) = H_1(\Gamma; \mathbb{Z}) \oplus H_1(F; \mathbb{Z})$.

(iii) The projection $q : G' \to G$ has a section if and only if there are elements $\mu_1, \dots, \mu_d, \nu_1, \dots, \nu_d$ of Γ such that the elements $\{\tilde{\alpha}_j = (\mu_j, \alpha_j)\}_{j=1}^d$ and $\{\tilde{\beta}_j = (\nu_j, \beta_j)\}_{j=1}^d$ of $\Gamma \times F$ satisfy $\prod_{j=1}^d [\tilde{\alpha}_j, \tilde{\beta}_j] \in \langle(\gamma, \delta)\rangle$. We have

$$\prod_{j=1}^d [\tilde{\alpha}_j, \tilde{\beta}_j] = \Big(\prod_{j=1}^d [\mu_j, \nu_j], \prod_{j=1}^d [\alpha_j, \beta_j] \Big) = \Big(\prod_{j=1}^d [\mu_j, \nu_j], \delta \Big).$$

By the argument in (i), the inclusion $(\prod_{j=1}^d [\mu_j, \nu_j], \delta) \in \langle(\gamma, \delta)\rangle$ implies that $\prod_{j=1}^d [\mu_j, \nu_j] = \gamma^r$ for some $r \in \mathbb{Z}$. Thus, both (γ, δ) and (γ^r, δ) lie in the group $\langle(\gamma, \delta)\rangle$. Then $(\gamma^{r-1}, 1)$ lies in this group. Claim (i) implies that $\gamma^{r-1} = 1$ so that $\prod_{j=1}^d [\mu_j, \nu_j] = \gamma$. Therefore, to prove (iii), it is enough to show that γ cannot be expanded as a product of d commutators in Γ for $d < n$. The formula $[x, y] \mapsto x \wedge y$ allows us to identify $[\Gamma, \Gamma]$ with the exterior square $H \wedge H$, where $H = H_1(\Gamma; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ is a $2n$-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. Set $H^* = \mathrm{Hom}(H, \mathbb{Z}/2\mathbb{Z})$. We can identify elements of $H \wedge H$ with symmetric bilinear forms $F : H^* \times H^* \to \mathbb{Z}/2\mathbb{Z}$ such that $F(x, x) = 0$ for all $x \in H^*$. It is easy to see that if an element of $[\Gamma, \Gamma]$ is a product of d commutators, then the rank of the corresponding bilinear form $H^* \times H^* \to \mathbb{Z}/2\mathbb{Z}$ is less than or equal to $2d$. The bilinear form corresponding to γ is the direct sum of n bilinear forms presented by the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The rank of this direct sum being equal to $2n$, the element γ of Γ cannot be a product of d commutators in Γ for $d < n$.

The arguments above show that q has a section for all $d \geq n$. This is compatible with Theorem 7.1.2 which guarantees the existence of a section for $d \geq n^2 - n/2$.

V.8 Applications of Theorem 1.2.1

We discuss several applications of Theorems 1.2.1 and 6.1. Throughout this section, W is a closed connected oriented surface of positive genus, $\pi = \pi_1(W)$, and K is an algebraically closed field of characteristic zero.

8.1 Enumeration of fiber bundles. The Frobenius–Mednykh formula (1.1.a) can be reformulated in terms of principal Γ-bundles over W. Recall that a principal Γ-bundle over W is a regular covering $\widetilde{W} \to W$ with group of automorphisms Γ; cf. Remark I.3.5.2. Let $\mathcal{P}(W, \Gamma)$ be the set of isomorphism classes of principal Γ-bundles over W. Any group homomorphism $g: \pi \to \Gamma$ determines a principal Γ-bundle ξ_g over W by pulling back the universal covering over $K(\Gamma, 1)$ along a map $W \to K(\Gamma, 1)$ inducing g in π_1. It is well known that every principal Γ-bundle over W is isomorphic to ξ_g for some g. Two homomorphisms $g_1, g_2: \pi \to \Gamma$ determine isomorphic principal Γ-bundles over W if and only if $g_1 = hg_2h^{-1}$ for some $h \in \Gamma$. Therefore

$$\mathcal{P}(W, \Gamma) = \operatorname{Hom}(\pi, \Gamma)/\Gamma,$$

where Γ acts on $\operatorname{Hom}(\pi, \Gamma)$ by conjugation. The stabilizer of $g \in \operatorname{Hom}(\pi, \Gamma)$ under this action is the group $\{h \in \Gamma \mid hgh^{-1} = g\}$ isomorphic to the group of automorphisms $\operatorname{Aut}(\xi_g)$ of ξ_g. Combining these facts, we obtain

$$|\operatorname{Hom}(\pi, \Gamma)| = \sum_{\xi \in \mathcal{P}(W,\Gamma)} |\Gamma|/|\operatorname{Aut}(\xi)|.$$

Formula (1.1.a) implies therefore that

$$\sum_{\xi \in \mathcal{P}(W,\Gamma)} 1/|\operatorname{Aut}(\xi)| = \sum_{\rho \in \operatorname{Irr}(\Gamma;K)} \big(|\Gamma|/\dim\rho\big)^{-\chi(W)}. \tag{8.1.a}$$

We view the left-hand side of this formula as the global measure of $\mathcal{P}(W, \Gamma)$ that counts the elements $\xi \in \mathcal{P}(W, \Gamma)$ with the weights $1/|\operatorname{Aut}(\xi)|$.

Theorem 1.2.1 yields a relative version of (8.1.a) associated with the epimorphism $q: G' \to G$. Fix a principal G-bundle ξ over W. By a *lift of* ξ *to a principal* G'-*bundle*, we mean a pair (a principal G'-bundle ξ' over W, an isomorphism of G-bundles $f: \xi'/\Gamma \cong \xi$). Here ξ'/Γ is the principal G-bundle over W obtained by factorizing the total space of ξ' by $\Gamma = \operatorname{Ker} q$. An *isomorphism* $(\xi'_1, f_1) \approx (\xi'_2, f_2)$ of two such lifts of ξ is an isomorphism of principal G'-bundles $\xi'_1 \to \xi'_2$ such that the induced isomorphism of principal G-bundles $\xi'_1/\Gamma \to \xi'_2/\Gamma$ composed with $f_2: \xi'_2/\Gamma \to \xi$ gives f_1. In particular, an *automorphism* of a lift (ξ', f) of ξ is an automorphism of ξ' inducing the identity on ξ'/Γ. Such automorphisms form a group denoted by $\operatorname{Aut}_\xi(\xi')$. The set of isomorphism classes of lifts of ξ to principal G'-bundles is denoted by $\mathcal{P}(\xi)$.

Fix a homomorphism $g: \pi \to G$ such that $\xi = \xi_g$. It is clear that every homomorphism $g' \in \operatorname{Hom}_g(\pi, G')$ determines a lift of ξ to a principal G'-bundle, and every such lift arises from some $g' \in \mathrm{Hom}_g(\pi, G')$. Two homomorphisms $g_1', g_2' \in \mathrm{Hom}_g(\pi, G')$ determine isomorphic lifts of ξ if and only if $g_1' = h g_2' h^{-1}$ for some $h \in \Gamma$. Therefore

$$\mathcal{P}(\xi) = \mathrm{Hom}_g(\pi, G')/\Gamma,$$

where Γ acts on $\mathrm{Hom}_g(\pi, G')$ by conjugation. The stabilizer of $g' \in \mathrm{Hom}_g(\pi, G')$ under this action is the group $\{h \in \Gamma \mid hg'h^{-1} = g'\}$ isomorphic to $\mathrm{Aut}_\xi(\xi_{g'})$. Combining these facts, we obtain

$$|\,\mathrm{Hom}_g(\pi, G')| = \sum_{\xi' \in \mathcal{P}(\xi)} |\Gamma|/|\,\mathrm{Aut}_\xi(\xi')|. \tag{8.1.b}$$

Theorem 1.2.1 implies that

$$\sum_{\xi' \in \mathcal{P}(\xi)} 1/|\,\mathrm{Aut}_\xi(\xi')| = \sum_{\substack{\rho \in \mathrm{Irr}(\Gamma; K) \\ G_\rho \supset g(\pi)}} \big(|\Gamma|/\dim \rho\big)^{-\chi(W)}\; g^*(\zeta_\rho)([W]). \tag{8.1.c}$$

Formula (8.1.b) and Corollary 1.3.3 imply that $\sum_{\xi' \in \mathcal{P}(\xi)} |\,\mathrm{Aut}_\xi(\xi')|^{-1}$ is a non-negative integer divisible by $|Z(\Gamma)|^{2d-2}$, where d is the genus of W. By Corollary 1.3.4, this integer is smaller than or equal to $|\Gamma|^{-1}\,|\,\mathrm{Hom}(\pi, \Gamma)|$.

Formula (8.1.c) implies that ξ lifts to a principal G'-bundle if and only if

$$\sum_{\substack{\rho \in \mathrm{Irr}(\Gamma; K) \\ G_\rho \supset g(\pi)}} (\dim \rho)^{\chi(W)}\; g^*(\zeta_\rho)([W]) \neq 0.$$

Theorems 7.1.1 and 7.1.2 show that in the case where the group Γ is abelian or the genus of W is bigger than $(1/2)\log_2(|[\Gamma, G']| - 1)$, the bundle ξ lifts to a principal G'-bundle if and only if the homology class $g_*([W]) \in H_2(g(\pi))$ lies in the image of the homomorphism $q_* \colon H_2(q^{-1}(g(\pi))) \to H_2(g(\pi))$.

Theorem 6.1 yields a generalization of (8.1.c). Pick a K^*-valued 2-cocycle θ on G'. For a principal G'-bundle ξ' on W, set

$$\theta_{\xi'} = (g')^*([\theta]) \in H^2(\pi; K^*),$$

where $[\theta] \in H^2(G'; K^*)$ is the cohomology class of θ and $g' \colon \pi \to G'$ is any homomorphism such that $\xi' = \xi_{g'}$. The cohomology class $\theta_{\xi'}$ does not depend on the choice of g' because the conjugations in G' act trivially on $H^*(G'; K^*)$. Theorem 6.1 implies that for the principal G-bundle ξ over W determined by a homomorphism $g \colon \pi \to G$,

$$\sum_{\xi' \in \mathcal{P}(\xi)} \theta_{\xi'}([W])/|\,\mathrm{Aut}_\xi(\xi')| = \sum_{\substack{\rho \in \mathrm{Irr}_\theta(\Gamma; K) \\ G_\rho \supset g(\pi)}} \big(|\Gamma|/\dim \rho\big)^{-\chi(W)}\; g^*(\zeta_\rho)([W]).$$

For $G = 1$, this formula boils down to

$$\sum_{\xi' \in \mathcal{P}(W,\Gamma)} \theta_{\xi'}([W])/|\,\mathrm{Aut}(\xi')| = \sum_{\rho \in \mathrm{Irr}_\theta(\Gamma; K)} \big(|\Gamma|/\dim \rho\big)^{-\chi(W)}.$$

8.2 Enumeration of sections. Theorem 1.2.1 may be used to count homotopy classes of sections of Serre fibrations (in particular, of locally trivial fiber bundles) over the surface W. Let $p: E \to W$ be a Serre fibration with fiber F. A *section* of p is a continuous mapping $s : W \to E$ such that $ps = \mathrm{id}_W$. Two sections of p are *homotopic* if they can be deformed into each other in the class of sections of p. We say that two sections $W \to E$ are obtained from each other by *bubbling* if they coincide on the complement of a small open disc $D \subset W$. The restrictions of such two sections on the closed disc $\overline{D} \subset W$ form then a mapping $S^2 \to E$, a "bubble". Two sections of p are *bubble equivalent* if they may be obtained from each other by a finite sequence of bubblings. Decomposing a deformation of a section into local deformations, one easily observes that homotopic sections are bubble equivalent. (If $\pi_2(F) = 0$, then the converse is also true so that the bubble equivalence is just the homotopy.) Denote the set of bubble equivalence classes of sections of p by $\mathcal{S}(p)$. Our aim is to count the elements of this set with certain weights; see formula (8.2.c) below.

The definition of $\mathcal{S}(p)$ has a pointed version as follows. Fix a base point $e \in E$ and set $w = p(e) \in W$. A section $s: W \to E$ of p is *pointed* if $s(w) = e$. Two pointed sections of p are *homotopic* if they can be deformed into each other in the class of pointed sections of p. The definitions of the bubbling and of the bubbling equivalence extend to pointed sections in the obvious way with the only difference that the disk D in the definition of a bubbling should lie in $W - \{w\}$. As above, homotopic pointed sections are bubble equivalent. The converse is true if $\pi_2(F) = 0$. We denote the set of bubble equivalence classes of pointed sections of p by $\mathcal{S}_*(p)$.

Suppose from now on that the fiber $F = p^{-1}(w)$ of p is path-connected and its fundamental group $\Phi = \pi_1(F, e)$ is finite. Set $\pi = \pi_1(W, w)$ and $\pi' = \pi_1(E, e)$. The exact homotopy sequence of p shows that the homomorphism $p_\#: \pi' \to \pi$ is surjective and $\operatorname{Ker} p_\# = \Phi$. By Sections 1–3, every irreducible (linear or projective) representation ρ of Φ over K determines a subgroup π_ρ of π and a cohomology class $\zeta_\rho \in H^2(\pi_\rho; K^*)$.

For any pointed section $s: W \to E$ of p, the induced homomorphism $s_\#: \pi \to \pi'$ is a section of $p_\#$ in the sense of Section 1.3. It is clear that $s_\#$ is preserved under the bubblings of s. It is easy to check that the resulting mapping

$$\mathcal{S}_*(p) \to S_*(p_\#), \quad s \mapsto s_\#, \tag{8.2.a}$$

is bijective. Thus, $|\mathcal{S}_*(p)| = |S_*(p)|$. Corollary 1.3.3 implies the following formula:

$$|\mathcal{S}_*(p)| = |\Phi| \sum_{\substack{\rho \in \operatorname{Irr}(\Phi; K) \\ \pi_\rho = \pi}} \left(|\Phi| / \dim \rho\right)^{-\chi(W)} \zeta_\rho([W]). \tag{8.2.b}$$

We now rewrite (8.2.b) in terms of non-pointed sections of p. Since F is path-connected, any section of p is homotopic to a pointed section. This shows that the natural mapping $\mathcal{S}_*(p) \to \mathcal{S}(p)$ is surjective. This mapping may be described in terms of an action of Φ on $\mathcal{S}_*(p)$ as follows. The group Φ acts on the set $S_*(p_\#)$ by conjugation. This defines an action of Φ on $\mathcal{S}_*(p)$ via the bijection (8.2.a). It is easy to see that the orbits of the latter action are precisely the preimages of elements of $\mathcal{S}(p)$ under the natural mapping $\mathcal{S}_*(p) \to \mathcal{S}(p)$. Thus, $\mathcal{S}(p) = \mathcal{S}_*(p)/\Phi$. For $s \in \mathcal{S}(p)$, let $\mathrm{Aut}(s) \subset \Phi$ be the stabilizer of an element of $\mathcal{S}_*(p)$ projecting to s. The group $\mathrm{Aut}(s)$ is well defined up to conjugation in Φ. If Φ is finite, then

$$|\mathcal{S}_*(p)| = \sum_{s\in\mathcal{S}(p)} |\Phi|/|\operatorname{Aut}(s)|.$$

Formula (8.2.b) may be rewritten as

$$\sum_{s\in\mathcal{S}(p)} 1/|\operatorname{Aut}(s)| = \sum_{\substack{\rho\in\mathrm{Irr}(\Phi;K)\\ \pi_\rho=\pi}} \big(|\Phi|/\dim\rho\big)^{-\chi(W)}\, \zeta_\rho([W]). \qquad (8.2.c)$$

Corollaries 1.3.3 and 1.3.4 imply that $\sum_{s\in\mathcal{S}(p)} |\operatorname{Aut}(s)|^{-1}$ is a non-negative integer divisible by $|Z(\Phi)|^{2d-2}$, where d is the genus of W, and that this integer is smaller than or equal to $|\Phi|^{-1}\,|\operatorname{Hom}(\pi,\Phi)|$.

Formula (8.2.c) allows one to detect whether or not the fibration p has a section. Namely, p has a section if and only if

$$\sum_{\substack{\rho\in\mathrm{Irr}(\Phi;K)\\ \pi_\rho=\pi}} (\dim\rho)^{\chi(W)}\, \zeta_\rho([W]) \neq 0.$$

Theorems 7.1.1 and 7.1.2 show that in the case where the group Φ is abelian or the genus of W is bigger than $(1/2)\log_2(|[\Phi,\pi']|-1)$, the fibration p has a section if and only if the homomorphism $p_*\colon H_2(E)\to H_2(W)$ is surjective. In the case of a trivial fibration, formula (8.2.b) amounts to computing the number of pointed homotopy classes of maps $W\to F$. In this case, all the cohomology classes ζ_ρ are trivial and formula (8.2.b) follows directly from the Frobenius–Mednykh formula (1.1.a).

Note one more application of formula (8.2.c). Consider a degree $n \geq 2$ map $f\colon W'\to W$, where W' is a closed connected oriented surface. The fibration p over W lifts to a Serre fibration $p' = f^*(p)$ over W' with the same fiber. Applying (8.2.c) to p', we obtain

$$\sum_{s\in\mathcal{S}(p')} 1/|\operatorname{Aut}(s)| = \sum_{\substack{\rho\in\mathrm{Irr}(\Phi;K)\\ \pi_\rho\supset f_\#(\pi')}} \big(|\Phi|/\dim\rho\big)^{-\chi(W')}\, (\zeta_\rho([W]))^n,$$

where $\pi' = \pi_1(W')$ and $f_\#$ is the homomorphism $\pi'\to\pi=\pi_1(W)$ induced by f. Here we use the equalities $\pi'_\rho = f_\#^{-1}(\pi_\rho)$ and $\zeta_\rho([W']) = (\zeta_\rho([W]))^n$. By Lemma 2.3.2, if n is divisible by $a_\rho \dim\rho$ for all $\rho\in\mathrm{Irr}(\Phi;K)$, then $(\zeta_\rho([W]))^n = 1$ for all ρ, so that p' necessarily has a section. In particular, if n is divisible by $|H_1(\Phi)|\,|\Phi/Z(\Phi)|$, then p' has a section.

Theorem 6.1 yields a generalization of (8.2.c) involving a cohomology class $\Theta\in H^2(E;K^*)$ whose evaluation on $\pi_2(E)$ is equal to $1\in K^*$. Such Θ is necessarily induced from a unique element of $H^2(\pi';K^*)$. We represent the latter by a K^*-valued

2-cocycle θ on $\pi' = \pi_1(E, e)$. Theorem 6.1 and the arguments above in this section imply the following formula:

$$\sum_{s \in \mathcal{S}(p)} s^*(\Theta)([W])/|\operatorname{Aut}(s)| = \sum_{\substack{\rho \in \operatorname{Irr}_\theta(\Phi; K) \\ \pi\rho = \pi}} \big(|\Phi|/\dim\rho\big)^{-\chi(W)} \zeta_\rho([W]).$$

The assumption $\Theta(\pi_2(E)) = 1$ ensures that $s^*(\Theta) \in H^2(W; K^*)$ is a bubble equivalence invariant of s so that its evaluation on $[W]$ is well defined.

8.3 Non-abelian cohomology of surfaces. Theorem 1.2.1 yields interesting information about 1-dimensional non-abelian cohomology of $\pi = \pi_1(W)$. We begin by recalling the definition of the 1-dimensional non-abelian cohomology of an arbitrary group Π, cf. [Se]. Fix a left action of Π on a group Φ, i.e., a homomorphism $\Pi \to \operatorname{Aut}\Phi$. A map $\alpha\colon \Pi \to \Phi$ is a *cocycle* if $\alpha(ab) = \alpha(a)\, a(\alpha(b))$ for all $a, b \in \Pi$. Here $a(\alpha(b)) \in \Phi$ is the result of the action of a on $\alpha(b)$. For example, the mapping $\Pi \to \{1\} \subset \Phi$ is a cocycle. The set of all cocycles $\Pi \to \Phi$ is denoted by $Z^1(\Pi; \Phi)$. The group Φ acts on $Z^1(\Pi; \Phi)$ by

$$(\varphi\alpha)(a) = \varphi\, \alpha(a)\, (a\varphi)^{-1}$$

for all $\varphi \in \Phi$, $\alpha \in Z^1(\Pi; \Phi)$, and $a \in \Pi$. The quotient set of this action is denoted by $H^1(\Pi; \Phi)$ and called the (nonabelian) cohomology of Π with coefficients in Φ. For $h \in H^1(\Pi; \Phi)$, let $\Phi_h \subset \Phi$ be the stabilizer of any cocycle representing h. The group Φ_h is determined by h up to conjugation in Φ.

If Π is finitely generated and Φ is finite, then both sets $Z^1(\Pi; \Phi)$ and $H^1(\Pi; \Phi)$ are finite. Put

$$\mathcal{M}(\Pi; \Phi) = \sum_{h \in H^1(\Pi; \Phi)} 1/|\Phi_h| \in \mathbb{Q}.$$

We view $\mathcal{M}(\Pi; \Phi)$ as the global measure of the set $H^1(\Pi; \Phi)$ that counts the elements h of this set with the weights $1/|\Phi_h|$. Since any $h \in H^1(\Pi; \Phi)$ can be represented by precisely $|\Phi|/|\Phi_h|$ cocycles,

$$\mathcal{M}(\Pi; \Phi) = |\Phi|^{-1}\, |Z^1(\Pi; \Phi)|.$$

The constructions of Sections 1 and 2 can be adapted to this setting as follows. With an irreducible linear representation $\rho\colon \Phi \to \mathrm{GL}_n(\mathbb{C})$ of Φ we associate the group $\Pi_\rho \subset \Pi$ consisting of all $a \in \Pi$ such that the representation $\varphi \mapsto \rho(a^{-1}\varphi)$ of Φ is equivalent to ρ. This means that there is a matrix $M_a \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho(a^{-1}\varphi) = M_a^{-1}\, \rho(\varphi)\, M_a$ for all $\varphi \in \Phi$. Then there is a family of non-zero complex numbers $\{\zeta_{a,b}\}_{a,b \in \Pi_\rho}$ such that $\zeta_{a,b}\, M_a\, M_b = M_{ab}$ for all $a, b \in \Pi_\rho$. This family is a 2-cocycle representing a well-defined cohomology class $\zeta_\rho \in H^2(\Pi_\rho; \mathbb{C}^*)$.

8.3.1 Theorem. *For any action of $\pi = \pi_1(W)$ on a finite group Φ,*

$$\mathcal{M}(\pi; \Phi) = \sum_{\substack{\rho \in \operatorname{Irr}(\Phi; \mathbb{C}) \\ \pi\rho = \pi}} \big(|\Phi|/\dim\rho\big)^{-\chi(W)} \zeta_\rho([W]). \tag{8.3.a}$$

Proof. Let π' be the set of all pairs $(\varphi \in \Phi, a \in \pi)$ with multiplication

$$(\varphi, a)(\varphi', a') = (\varphi\,(a\varphi'), aa').$$

It is easy to check that π' is a group. The formula $p(\varphi, a) = a$ defines an epimorphism $p\colon \pi' \to \pi$ with kernel $\{(\varphi, 1)\}_{\varphi\in\Phi} = \Phi$. Every cocycle $\alpha\colon \pi \to \Phi$ defines a section s_α of p by $s_\alpha(a) = (\alpha(a), a)$ for $a \in \pi$. The formula $\alpha \mapsto s_\alpha$ establishes a bijection between the set $Z^1(\pi; \Phi)$ and the set $S_*(p)$ of the sections of p. Therefore

$$\mathcal{M}(\pi; \Phi) = |\Phi|^{-1}\,|Z^1(\pi; \Phi)| = |\Phi|^{-1}\,|S_*(p)|. \tag{8.3.b}$$

It remains to apply Corollary 1.3.5 and to observe that the definitions of π_ρ and ζ_ρ given in Sections 1 and 2 are equivalent in the present setting to the definitions given before the statement of the theorem. (The key point is that every $a \in \pi$ has a canonical lift $(1, a)$ to π' and

$$(1, a)^{-1}(\varphi, 1)(1, a) = (a^{-1}\varphi, 1)$$

for all $\varphi \in \Phi$.) □

Formula (8.3.b) and the remarks after Corollary 1.3.5 imply that $\mathcal{M}(\pi; \Phi)$ is a non-negative integer divisible by $|Z(\Phi)|^{2d-2}$, where d is the genus of W, and

$$\mathcal{M}(\pi; \Phi) \le |\Phi|^{-1}\,|\operatorname{Hom}(\pi, \Phi)|.$$

V.9 Further applications of Theorem 1.2.1

We discuss miscellaneous algebraic notions and results suggested by Theorem 1.2.1.

9.1 Extremal homology classes. We call a homology class $h \in H_2(G; \mathbb{Z})$ *extremal* (with respect to the given epimorphism $q\colon G' \to G$) if $\zeta_\rho(h) = 1$ for all irreducible complex linear representations ρ of $\Gamma = \operatorname{Ker} q$ such that $G_\rho = G$. For example, the zero homology class $h = 0$ is extremal. It is clear that the extremal homology classes form a subgroup of $H_2(G; \mathbb{Z})$.

For each $k \ge 1$, the function $v_k : H_2(G; \mathbb{Z}) \to \mathbb{Z}$ introduced in Section 7.2 takes on all extremal classes the same value which is the maximal value of v_k. In particular, if $h \in H_2(G; \mathbb{Z})$ is extremal, then $v_1(h) = v_1(0) > 0$. By Lemma 7.2.1(a), all extremal homology classes lie in $Q = \operatorname{Im}(q_*\colon H_2(G'; \mathbb{Z}) \to H_2(G; \mathbb{Z}))$.

Lemma 2.3.2 allows us to construct extremal homology classes as follows. Let a_Γ be the least common multiple of the numbers $(\dim \rho)\, a_\rho$, where ρ runs over $\operatorname{Irr}(\Gamma; \mathbb{C})$ and a_ρ is the integer defined in Lemma 2.3.2. By this lemma, $a_\Gamma\, \zeta_\rho = 0$ for all ρ. Therefore, all elements of $a_\Gamma\, H_2(G; \mathbb{Z})$ are extremal.

Similarly, let b_Γ be the least common multiple of the numbers $\dim \rho$, where ρ runs over $\operatorname{Irr}(\Gamma; \mathbb{C})$. By Lemma 2.3.3, all elements of $b_\Gamma Q$ are extremal.

9.2 Quasi-epimorphisms. A homomorphism g from a group Π to G is a *quasi-epimorphism* (with respect to $q\colon G' \to G$) if

$$g(\Pi) \cap (G - G_\rho) \neq \emptyset$$

for all $\rho \in \operatorname{Irr}(\Gamma;\mathbb{C})$ such that $G_\rho \neq G$. Clearly, all epimorphisms $\Pi \to G$ are quasi-epimorphisms. If $G_\rho = G$ for all ρ, then all homomorphisms $\Pi \to G$ are quasi-epimorphisms.

Let W be a closed connected oriented surface with fundamental group π. By Theorem 1.2.1, for a quasi-epimorphism $g\colon \pi \to G$,

$$|\operatorname{Hom}_g(\pi, G')| = |\Gamma| \sum_{\substack{\rho\in\operatorname{Irr}(\Gamma;\mathbb{C})\\ G_\rho = G}} \big(|\Gamma|/\dim\rho\big)^{-\chi(W)}\,\zeta_\rho([W]). \tag{9.2.a}$$

Therefore

$$|\operatorname{Hom}_g(\pi, G')| \leq |\Gamma| \sum_{\substack{\rho\in\operatorname{Irr}(\Gamma;\mathbb{C})\\ G_\rho = G}} \big(|\Gamma|/\dim\rho\big)^{-\chi(W)}. \tag{9.2.b}$$

This inequality is an equality if and only if $g_*([W]) \in H_2(G;\mathbb{Z})$ is an extremal homology class in the sense of Section 9.1. If $g\colon \pi \to G$ is a quasi-epimorphism and $g_*([W]) \in H_2(G;\mathbb{Z})$ is an extremal homology class, then (9.2.a) implies that g necessarily lifts to G'.

9.3 The genus norm. As was already mentioned, for any $h \in H_2(G;\mathbb{Z})$, there are a closed connected oriented surface W of positive genus and a homomorphism $g\colon \pi_1(W) \to G$ such that $g_*([W]) = h$. For $h \neq 0$, denote the minimal genus of such a surface by $|h|$. For $h = 0$, set $|h| = 0$. Clearly, $|-h| = |h| \geq 0$ for all h, and $|h| = 0$ if and only if $h = 0$. Taking connected sums of surfaces, we obtain that $|h + h'| \leq |h| + |h'|$ for all $h, h' \in H_2(G;\mathbb{Z})$. We call the mapping $H_2(G;\mathbb{Z}) \to \mathbb{Z}$, $h \mapsto |h|$ the *genus norm*. A computation of this norm is an interesting and largely open problem.

The functions $v_1, v_2, \ldots\colon H_2(G;\mathbb{Z}) \to \mathbb{Z}$ derived in Section 7.2 from the epimorphism $q\colon G' \to G$ may help to estimate the genus norm from below. Define a mapping $v\colon H_2(G;\mathbb{Z}) \times \mathbb{Z} \to \mathbb{Q}$ by

$$v(h,n) = \sum_{k\geq 1} v_k(h)/k^{2n} = v_1(h) + \sum_{k\geq 2} v_k(h)/k^{2n},$$

where $h \in H_2(G;\mathbb{Z})$ and $n \in \mathbb{Z}$. The mapping v is well defined because $v_k = 0$ for all sufficiently big k. Claims (a) and (e) of Lemma 7.2.1 imply that, given $h \in H_2(G;\mathbb{Z})$, we have $v(h,n) \geq 0$ for all sufficiently big n. Denote by $\langle h, q\rangle$ the minimal non-negative integer N such that $v(h,n) \geq 0$ for all $n \geq N$.

9.3.1 Lemma. *For any epimorphism $q\colon G' \to G$ with kernel Γ such that $G_\rho = G$ for all $\rho \in \operatorname{Irr}(\Gamma;\mathbb{C})$ and for any non-zero $h \in H_2(G;\mathbb{Z})$, we have $|h| \geq \langle h, q\rangle + 1$.*

Proof. Pick a non-zero $h \in H_2(G;\mathbb{Z})$. We must show that $v(h,n) \geq 0$ for all $n \geq |h| - 1$. Any closed connected oriented surface W of genus $n + 1 \geq |h|$ admits a degree one map onto a closed connected oriented surface of genus $|h|$. Therefore there is a homomorphism $g\colon \pi_1(W) \to G$ such that $g_*([W]) = h$. Theorem 1.2.1 and the assumption $G_\rho = G$ for all ρ give

$$|\operatorname{Hom}_g(\pi_1(W), G')| = |\Gamma|^{1-\chi(W)}\, v\big(h, -\tfrac{\chi(W)}{2}\big) = |\Gamma|^{2n+1}\, v(h,n).$$

Hence, $v(h,n) \geq 0$. □

Varying $q\colon G' \to G$ in the class of group epimorphisms with target G and finite kernel satisfying the conditions of this lemma, we obtain a family of estimates for the genus norm on $H_2(G;\mathbb{Z})$. The author does not know whether these estimates may be non-trivial. Explicit computations and examples would be welcome.

9.4 Free actions of groups on surfaces and the genus. Suppose temporarily that the group G is finite and consider a fixed-point-free, orientation preserving action of G on a closed connected oriented surface $\widetilde{W}$. This action determines an element of $H_2(G;\mathbb{Z})$ as follows. The quotient space $W = \widetilde{W}/G$ is a closed connected oriented surface. The projection $\widetilde{W} \to W$ is a covering that determines a homomorphism $g\colon \pi_1(W) \to G$. We associate with the action of G on $\widetilde{W}$ the class $g_*([W]) \in H_2(G;\mathbb{Z})$. This construction suggests the following minimal genus problem. Given $h \in H_2(G;\mathbb{Z})$, find the minimal integer m such that there is a fixed-point-free, orientation preserving action of G on a closed connected oriented surface of genus m with the associated homology class h.

Observe that the homomorphism $g\colon \pi_1(W) \to G$ above is onto, since $\widetilde{W}$ is connected. Conversely, any epimorphism $\pi_1(W) \to G$ defines a covering $\widetilde{W}$ of W and a fixed-point-free, orientation preserving action of G on $\widetilde{W}$. The genera $d, \tilde{d}$ of $W, \widetilde{W}$ are related by

$$2 - 2\tilde{d} = \chi(\widetilde{W}) = |G|\,\chi(W) = |G|(2 - 2d).$$

Thus, instead of free actions of G on surfaces of minimal genus, we can as well search for surfaces of minimal genus W admitting epimorphisms $\pi_1(W) \to G$ that realize h. In the latter formulation of the problem, the finiteness of G becomes irrelevant. It suffices to assume that G is finitely generated. For more on this and related problems see [CN], [Li], [Zi], and references therein.

To state our results, we introduce the following notation. For a finitely generated group G and any $h \in H_2(G;\mathbb{Z})$, let $\mu(h)$ denote the minimal positive integer μ such that there are a closed connected oriented surface W of genus μ and an epimorphism $g\colon \pi_1(W) \to G$ with $g_*([W]) = h$. Clearly, $\mu(h + h') \leq \mu(h) + \mu(h')$ for all $h, h' \in H_2(G;\mathbb{Z})$, but μ is not a norm because $\mu(0) > 0$. Note also that $\mu(h) \geq |h|$ for all h. The proof of Lemma 9.3.1 applies in this setting and gives

$$\mu(h) \geq \langle h, q \rangle + 1$$

for all $h \in H_2(G;\mathbb{Z})$ and all epimorphisms $q\colon G' \to G$ with finite kernel. No assumptions on the representations of the kernel are needed here.

9.5 Lifts of maps. Let $p\colon E \to X$ be a Serre fibration over a topological space X. Given a map f from a closed connected oriented surface W to X, one may be interested in the existence and the number of lifts of f to E. By a *lift* of f to E, we mean a map $f'\colon W \to E$ such that $pf' = f$. For $X = W$ and $f = \mathrm{id}_W$, we recover the setting of Section 8.2. All definitions and results of Section 8.2 extend to arbitrary p and f with the obvious changes. The key observation is that the lifts of f to E bijectively correspond to the sections of the induced Serre fibration $f^*(p)$ over W. We leave the details to the reader.

Chapter VI

Crossed G-categories and invariants of links

VI.1 G-categories

We introduce monoidal G-categories which will be our main algebraic tools in constructing invariants of colored G-links in the 3-dimensional sphere S^3. By a category, we mean a small category.

1.1 Generalities on monoidal categories. Let $\mathcal{C}$ be a monoidal category with unit object $\mathbb{1}$. We will write $U \in \mathcal{C}$ to indicate that U is an object of $\mathcal{C}$. Recall (see for instance [Mac]) that we have invertible *associativity morphisms* (also called *associativity constraints*)

$$\{a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)\}_{U,V,W \in \mathcal{C}} \tag{1.1.a}$$

and invertible morphisms (called *unit constraints*)

$$\{l_U : U \to U \otimes \mathbb{1}, \quad r_U : U \to \mathbb{1} \otimes U\}_{U \in \mathcal{C}} \tag{1.1.b}$$

satisfying the *pentagon identity*

$$(\mathrm{id}_U \otimes a_{V,W,X})\, a_{U,V \otimes W,X}\, (a_{U,V,W} \otimes \mathrm{id}_X) = a_{U,V,W \otimes X}\, a_{U \otimes V,W,X} \tag{1.1.c}$$

and the *triangle identity*

$$a_{U,\mathbb{1},V}(l_U \otimes \mathrm{id}_V) = \mathrm{id}_U \otimes r_V \tag{1.1.d}$$

for any $U, V, W, X \in \mathcal{C}$. The morphisms l, r should satisfy $l_{\mathbb{1}} = r_{\mathbb{1}}$ and be natural in the sense that for any morphism $f : U \to V$ we have $l_V f = (f \otimes \mathrm{id}_{\mathbb{1}})l_U$ and $r_V f = (\mathrm{id}_{\mathbb{1}} \otimes f)r_U$. The associativity morphisms (1.1.a) should be natural in a similar sense.

A *left duality* in $\mathcal{C}$ associates to any object $U \in \mathcal{C}$ an object $U^* \in \mathcal{C}$ and two morphisms

$$b_U : \mathbb{1} \to U \otimes U^*, \quad d_U : U^* \otimes U \to \mathbb{1} \tag{1.1.e}$$

such that

$$(\mathrm{id}_U \otimes d_U)\, a_{U,U^*,U}\, (b_U \otimes \mathrm{id}_U) r_U = l_U, \tag{1.1.f}$$

$$(d_U \otimes \mathrm{id}_{U^*})\, a^{-1}_{U^*,U,U^*} (\mathrm{id}_{U^*} \otimes b_U) l_{U^*} = r_{U^*}. \tag{1.1.g}$$

We call the morphisms (1.1.a), (1.1.b), (1.1.e) the *structural morphisms* of $\mathcal{C}$.

A monoidal category $\mathcal{C}$ is *strict* if for any $U, V, W \in \mathcal{C}$,

$$(U \otimes V) \otimes W = U \otimes (V \otimes W) \quad \text{and} \quad U = U \otimes \mathbb{1} = \mathbb{1} \otimes U$$

and the morphisms $\{a_{U,V,W}\}_{U,V,W \in \mathcal{C}}$, $\{l_U, r_U\}_{U \in \mathcal{C}}$ are the identity morphisms. For a strict monoidal category $\mathcal{C}$, Formulas (1.1.f), (1.1.g) simplify to

$$(\mathrm{id}_U \otimes d_U)(b_U \otimes \mathrm{id}_U) = \mathrm{id}_U, \quad (d_U \otimes \mathrm{id}_{U^*})(\mathrm{id}_{U^*} \otimes b_U) = \mathrm{id}_{U^*}. \tag{1.1.h}$$

It is well known that each monoidal category is equivalent to a strict monoidal category.

1.2 *G*-categories. A monoidal category $\mathcal{C}$ is *K-additive* if the Hom-sets in $\mathcal{C}$ are modules over the ring K and both the composition and the tensor product of morphisms are bilinear over K. We say that a K-additive category $\mathcal{C}$ *splits as a disjoint union of subcategories* $\{\mathcal{C}_\alpha\}$ labeled by certain indices α if

- each $\mathcal{C}_\alpha$ is a full subcategory of $\mathcal{C}$;
- each object of $\mathcal{C}$ belongs to $\mathcal{C}_\alpha$ for a unique α;
- $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$ with $\alpha \neq \beta$ implies that $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$.

For a group G, a *G-category over K* is a K-additive monoidal category with left duality $\mathcal{C}$ that splits as a disjoint union of subcategories $\{\mathcal{C}_\alpha\}$ labeled by $\alpha \in G$ such that

(i) $\mathbb{1} \in \mathcal{C}_1$ and if $U \in \mathcal{C}_\alpha$, $V \in \mathcal{C}_\beta$, then $U \otimes V \in \mathcal{C}_{\alpha\beta}$;
(ii) if $U \in \mathcal{C}_\alpha$, then $U^* \in \mathcal{C}_{\alpha^{-1}}$.

We shall write $\mathcal{C} = \coprod_\alpha \mathcal{C}_\alpha$ and call the subcategories $\{\mathcal{C}_\alpha\}$ of $\mathcal{C}$ the *components* of $\mathcal{C}$. The category $\mathcal{C}_1$ corresponding to the neutral element $1 \in G$ is called the *neutral component* of $\mathcal{C}$. Conditions (i) and (ii) show that $\mathcal{C}_1$ is closed under tensor multiplication and taking the dual object. Thus, $\mathcal{C}_1$ is a K-additive monoidal category with left duality.

1.3 Example: *G*-categories from 3-cocycles. It is well known that 3-cocycles give rise to associativity morphisms in categories. Here we adapt this construction to our context. Let $a = \{a_{\alpha,\beta,\gamma} \in K^*\}_{\alpha,\beta,\gamma \in G}$ be a 3-cocycle of the group G with values in the multiplicative group K^* of invertible elements of K. Thus

$$a_{\alpha\beta,\gamma,\delta}\, a_{\alpha,\beta,\gamma\delta} = a_{\alpha,\beta,\gamma}\, a_{\alpha,\beta\gamma,\delta}\, a_{\beta,\gamma,\delta} \tag{1.3.a}$$

for any $\alpha, \beta, \gamma, \delta \in G$. Let $b = \{b_\alpha \in K^*\}_{\alpha \in G}$ be a family of elements of K^*. With the pair (a, b) we associate a G-category $\mathcal{C}$ as follows. For $\alpha \in G$, we define $\mathcal{C}_\alpha$ to be a category with one object V_α. For $\alpha, \beta \in G$, set

$$\mathrm{Hom}(V_\alpha, V_\beta) = \begin{cases} K & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

The tensor product is given by $V_\alpha \otimes V_\beta = V_{\alpha\beta}$. The composition and tensor product of morphisms is given by multiplication in K. Clearly, $\mathbb{1} = V_1$ is the unit object of $\mathcal{C}$. Now we define the structural morphisms in $\mathcal{C}$. The associativity morphism (1.1.a) for $U = V_\alpha, V = V_\beta, W = V_\gamma$ is

$$a_{\alpha,\beta,\gamma} \in K^* \subset K = \operatorname{End}(V_{\alpha\beta\gamma}).$$

The pentagon identity follows from (1.3.a). The morphisms $l_\alpha \colon V_\alpha \to V_\alpha \otimes \mathbb{1} = V_\alpha$ and $r_\alpha \colon V_\alpha \to \mathbb{1} \otimes V_\alpha = V_\alpha$ are defined by

$$l_\alpha = a_{\alpha,1,1}^{-1} \in K^* \subset K = \operatorname{End}(V_\alpha), \quad r_\alpha = a_{1,1,\alpha} \in K^* \subset K = \operatorname{End}(V_\alpha).$$

The triangle identity follows from the equality $a_{\alpha,1,\beta} = a_{\alpha,1,1}\, a_{1,1,\beta}$ obtained from (1.3.a) by the substitution $\beta = 1, \gamma = 1, \delta = \beta$. The dual of V_α is by definition $V_{\alpha^{-1}}$. The duality morphism $\mathbb{1} \to V_\alpha \otimes V_{\alpha^{-1}} = \mathbb{1}$ is defined to be $b_\alpha \in K^* \subset K = \operatorname{End}(\mathbb{1})$. The morphism $d_\alpha \colon V_{\alpha^{-1}} \otimes V_\alpha \to \mathbb{1}$ is determined from (1.1.f) or from (1.1.g). In fact these equations give two different expressions for d_α:

$$d_\alpha = b_\alpha^{-1} \left(a_{\alpha,\alpha^{-1},\alpha}\, a_{\alpha,1,1}\, a_{1,1,\alpha}\right)^{-1}, \tag{1.3.b}$$

$$d_\alpha = b_\alpha^{-1}\, a_{\alpha^{-1},\alpha,\alpha^{-1}}\, a_{\alpha^{-1},1,1}\, a_{1,1,\alpha^{-1}}. \tag{1.3.c}$$

To show that the right-hand sides are equal we substitute $\beta = \delta = \alpha^{-1}, \gamma = \alpha$ in (1.3.a). This gives

$$\begin{aligned} a_{1,\alpha,\alpha^{-1}}\, a_{\alpha,\alpha^{-1},1} &= a_{\alpha,\alpha^{-1},\alpha}\, a_{\alpha,1,\alpha^{-1}}\, a_{\alpha^{-1},\alpha,\alpha^{-1}} \\ &= a_{\alpha,\alpha^{-1},\alpha}\, a_{\alpha,1,1}\, a_{1,1,\alpha^{-1}}\, a_{\alpha^{-1},\alpha,\alpha^{-1}}. \end{aligned} \tag{1.3.d}$$

Substituting $\alpha = \gamma = 1$ in (1.3.a) we obtain that $a_{1,\alpha,1} = 1$ for all α. Substituting $\gamma = \delta = 1, \beta = \alpha^{-1}$ (resp. $\alpha = 1, \delta = \beta = \gamma^{-1}$) in (1.3.a) we obtain that $a_{\alpha,\alpha^{-1},1} = a_{\alpha^{-1},1,1}^{-1}$ (resp. $a_{1,\alpha,\alpha^{-1}} = a_{1,1,\alpha}^{-1}$). Substituting the latter equalities in (1.3.d) we obtain that the right-hand sides of (1.3.b) and (1.3.c) are equal. It is clear that $\mathcal{C}$ satisfies all axioms of a G-category.

1.4 Operations on group-categories. We define several simple operations on group-categories. Note first that the group-categories can be pulled back along group homomorphisms. Having a group homomorphism $q \colon G' \to G$, we can derive from any G-category $\mathcal{C}$ a G'-category $\mathcal{C}' = q^*(\mathcal{C})$ such that $\mathcal{C}'_\alpha = \mathcal{C}_{q(\alpha)}$ for any $\alpha \in G'$. Composition, tensor multiplication, and the structural morphisms in $\mathcal{C}$ induce the corresponding operations in $\mathcal{C}'$ in the obvious way.

The group-categories can be pushed forward along group homomorphisms. Having a group homomorphism $q \colon G' \to G$, we can derive from any G'-category $\mathcal{C}'$ a G-category $\mathcal{C} = q_*(\mathcal{C}')$ by $\mathcal{C}_\alpha = \coprod_{\beta \in q^{-1}(\alpha)} \mathcal{C}'_\beta$ for any $\alpha \in G$. Composition, tensor multiplication, and the structural morphisms in $\mathcal{C}'$ induce the corresponding operations in $\mathcal{C}$ in the obvious way.

For any family of G-categories $\{\mathcal{C}^i\}_{i\in I}$, we define a *direct product* $\mathcal{C} = \times_i \mathcal{C}^i$. The category $\mathcal{C}$ is a disjoint union of categories $\{\mathcal{C}_\alpha\}_{\alpha\in G}$. The objects of $\mathcal{C}_\alpha$ are families $\{U_i \in \mathcal{C}^i_\alpha\}_{i\in I}$. The operations on the objects and the unit object are defined by

$$\{U_i\}_i \otimes \{U'_i\}_i = \{U_i \otimes U'_i\}_i, \quad (\{U_i\}_i)^* = \{U_i^*\}_i, \quad \mathbb{1}_{\mathcal{C}} = \{\mathbb{1}_{\mathcal{C}^i}\}_i, \tag{1.4.a}$$

where i runs over I. A morphism $\{U_i \in \mathcal{C}^i\}_i \to \{U'_i \in \mathcal{C}^i\}_i$ in $\mathcal{C}$ is a family $\{f_i : U_i \to U'_i\}_i$ where each f_i is a morphism in $\mathcal{C}^i$. Thus

$$\mathrm{Hom}_{\mathcal{C}}(\{U_i\}_i, \{U'_i\}_i) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}^i}(U_i, U'_i).$$

The composition of morphisms is coordinate-wise, i.e.,

$$\{f'_i : U'_i \to U''_i\} \circ \{f_i : U_i \to U'_i\} = \{f'_i f_i : U_i \to U''_i\}.$$

The K-additive structure on $\mathcal{C}$, the tensor product for morphisms and the structural morphisms are defined coordinate-wise. All the axioms of a G-category follow from the fact that they are satisfied coordinate-wise.

For a finite family of G-categories $\{\mathcal{C}^i\}_{i\in I}$, we define a *tensor product* $\mathcal{C}' = \otimes_i \mathcal{C}^i$. The category $\mathcal{C}'$ is a disjoint union of categories $\{\mathcal{C}'_\alpha\}_{\alpha\in G}$. The objects of $\mathcal{C}'_\alpha$ are the same as the objects of the category $\mathcal{C}_\alpha \subset \times_i \mathcal{C}^i$ above. The operations on the objects and the unit object are defined by (1.4.a). By definition,

$$\mathrm{Hom}_{\mathcal{C}'}(\{U_i\}_i, \{U'_i\}_i) = \bigotimes_{i\in I} \mathrm{Hom}_{\mathcal{C}^i}(U_i, U'_i).$$

This K-module is additively generated by the vectors of type $\otimes_i (f_i : U_i \to U'_i)$. The composition of morphisms is defined on the generators by

$$\bigotimes_i (f'_i : U'_i \to U''_i) \circ \bigotimes_i (f_i : U_i \to U'_i) = \bigotimes_i (f'_i f_i : U_i \to U''_i).$$

This extends to arbitrary morphisms by K-linearity and turns $\mathcal{C}'$ into a K-additive category. The tensor product of morphisms is defined on the generators by $(\otimes_i f_i) \otimes (\otimes_i g_i) = \otimes_i (f_i \otimes g_i)$ and extends to arbitrary morphisms by K-linearity. Observe that there is a canonical functor $\times_i \mathcal{C}^i \to \mathcal{C}'$ which is the identity on the objects and carries a morphism $\{f_i\}_i$ in $\times_i \mathcal{C}^i$ into the morphism $\bigotimes_i f_i$ in $\mathcal{C}'$. This functor is by no means K-linear but does preserve the tensor product. Applying this functor to the structural morphisms in $\times_i \mathcal{C}^i$ defined above we obtain structural morphisms in $\mathcal{C}'$ satisfying all the conditions of Section 1.1. In this way $\mathcal{C}'$ becomes a G-category.

VI.2 Crossed, braided, and ribbon G-categories

We discuss natural additional structures on G-categories: crossed actions of G, braidings, and twists.

2.1 Crossed G-categories. Let $\mathcal{C}$ be a K-additive monoidal category with left duality. By an *automorphism* of $\mathcal{C}$ we mean an invertible K-linear (on the morphisms) functor $\varphi\colon \mathcal{C} \to \mathcal{C}$ which preserves the tensor product, the unit object, the duality, and the structural morphisms a, l, r, b, d. Thus,

$$\varphi(\mathbb{1}) = \mathbb{1}, \quad \varphi(U \otimes V) = \varphi(U) \otimes \varphi(V), \quad \varphi(U^*) = (\varphi(U))^*,$$
$$\varphi(a_{U,V,W}) = a_{\varphi(U),\varphi(V),\varphi(W)},$$
$$\varphi(l_U) = l_{\varphi(U)}, \quad \varphi(r_U) = r_{\varphi(U)}, \quad \varphi(b_U) = b_{\varphi(U)}, \quad \varphi(d_U) = d_{\varphi(U)}$$

for any objects $U, V, W \in \mathcal{C}$ and $\varphi(f \otimes g) = \varphi(f) \otimes \varphi(g)$ for any morphisms f, g in $\mathcal{C}$. The group of automorphisms of $\mathcal{C}$ is denoted by $\operatorname{Aut}(\mathcal{C})$.

A *crossed G-category over K* is a G-category $\mathcal{C}$ endowed with a group homomorphism $\varphi\colon G \to \operatorname{Aut}(\mathcal{C})$ such that the functor $\varphi_\alpha = \varphi(\alpha)\colon \mathcal{C} \to \mathcal{C}$ maps $\mathcal{C}_\beta$ into $\mathcal{C}_{\alpha\beta\alpha^{-1}}$ for all $\alpha, \beta \in G$. For any objects $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$, set

$$^{U}V = \varphi_\alpha(V) \in \mathcal{C}_{\alpha\beta\alpha^{-1}}.$$

In particular, $^{U}U = \varphi_\alpha(U) \in \mathcal{C}_\alpha$ for any $U \in \mathcal{C}_\alpha$. Note the identities

$$^{U}(V \otimes W) = {}^{U}V \otimes {}^{U}W \tag{2.1.a}$$
$$^{(U\otimes V)}W = {}^{U}({}^{V}W), \tag{2.1.b}$$
$$^{U}(V^*) = ({}^{U}V)^*, \tag{2.1.c}$$
$$^{\mathbb{1}}V = {}^{U}({}^{U^*}V) = {}^{U^*}({}^{U}V) = V, \quad {}^{U}\mathbb{1} = \mathbb{1}, \tag{2.1.d}$$

for any $U, V, W \in \mathcal{C}$. Similarly, for an object $U \in \mathcal{C}_\alpha$ and a morphism $f\colon V \to V'$ in $\mathcal{C}$ set

$$^{U}f = \varphi_\alpha(f)\colon {}^{U}V \to {}^{U}(V').$$

Note the identities

$$^{U}(f' \circ f) = {}^{U}(f') \circ {}^{U}f, \tag{2.1.e}$$
$$^{U}(f \otimes g) = {}^{U}f \otimes {}^{U}g, \tag{2.1.f}$$

$$\begin{gathered} ^{U}(\mathrm{id}_V) = \mathrm{id}_{({}^{U}V)}, \quad {}^{U}(a_{V,W,X}) = a_{{}^{U}V,{}^{U}W,{}^{U}X}, \\ ^{U}(l_V) = l_{({}^{U}V)}, \quad {}^{U}(r_V) = r_{({}^{U}V)}, \quad {}^{U}(b_V) = b_{({}^{U}V)}, \quad {}^{U}(d_V) = d_{({}^{U}V)}, \end{gathered} \tag{2.1.g}$$

$$^{(U\otimes V)}f = {}^{U}({}^{V}f), \quad {}^{\mathbb{1}}f = {}^{U}({}^{U^*}f) = {}^{U^*}({}^{U}f) = f. \tag{2.1.h}$$

Examples of crossed G-categories will be given in Section 2.6 and in further sections.

Many notions of the theory of categories naturally extend to (crossed) G-categories. For instance, one can consider semisimple G-categories, fusion G-categories, etc. We shall focus here on braidings in crossed G-categories.

The notion of a crossed G-category may be generalized by replacing the automorphisms φ of $\mathcal{C}$ by monoidal equivalences $\mathcal{C} \to \mathcal{C}$. In particular, instead of the equality $\varphi(U \otimes V) = \varphi(U) \otimes \varphi(V)$, one may require a natural isomorphism $\varphi(U \otimes V) \to \varphi(U) \otimes \varphi(V)$ satisfying appropriate coherence conditions. Similarly, the equality (2.1.b) may be replaced by a natural isomorphism. Such generalizations may be interesting from the topological viewpoint but we shall not study them here.

2.2 Braiding in G-categories. Let $\mathcal{C}$ be a crossed G-category. A *braiding* in $\mathcal{C}$ is a system of invertible morphisms

$$\{c_{U,V} \colon U \otimes V \to {}^{U}V \otimes U\}_{U,V \in \mathcal{C}} \tag{2.2.a}$$

satisfying the following three conditions:

(2.2.1) for any morphisms $f \colon U \to U'$, $g \colon V \to V'$ such that U, U' lie in the same component of $\mathcal{C}$,

$$c_{U',V'}(f \otimes g) = ({}^{U}g \otimes f)\, c_{U,V}; \tag{2.2.b}$$

(2.2.2) for any objects $U, V, W \in \mathcal{C}$,

$$c_{U\otimes V,W} = a_{({}^{U\,V})W,U,V}\,(c_{U,{}^{V}W} \otimes \mathrm{id}_V)\, a^{-1}_{U,{}^{V}W,V}\,(\mathrm{id}_U \otimes c_{V,W})\, a_{U,V,W}, \tag{2.2.c}$$

$$c_{U,V\otimes W} = a^{-1}_{{}^{U}V,{}^{U}W,U}\,(\mathrm{id}_{({}^{U}V)} \otimes c_{U,W})\, a_{{}^{U}V,U,W}\,(c_{U,V} \otimes \mathrm{id}_W)\, a^{-1}_{U,V,W}; \tag{2.2.d}$$

(2.2.3) the action of G on $\mathcal{C}$ preserves the braiding, i.e., for any $\alpha \in G$ and any $V, W \in \mathcal{C}$,

$$\varphi_\alpha(c_{V,W}) = c_{\varphi_\alpha(V),\varphi_\alpha(W)}.$$

Note that if in (2.2.1) the objects U, U' do not lie in the same component of $\mathcal{C}$ then both sides of (2.2.b) are equal to 0 and have the same source $U \otimes V$ but may have different targets. Formulas (2.1.a) and (2.1.b) imply that the targets of the morphisms on the left- and right-hand sides of (2.2.c) and (2.2.d) are the same so that these equalities make sense.

A crossed G-category endowed with a braiding is said to be *braided*. For $G = 1$, we obtain the standard definition of a braided monoidal category.

A braiding in a crossed G-category $\mathcal{C}$ satisfies a version of the Yang–Baxter identity. Assume for simplicity that $\mathcal{C}$ is strict. Then for any braiding (2.2.a) in $\mathcal{C}$ and any objects $U, V, W \in \mathcal{C}$,

$$\begin{aligned}(c_{{}^{U}V,{}^{U}W} \otimes \mathrm{id}_U)\,(\mathrm{id}_{({}^{U}V)} \otimes c_{U,W})\,(c_{U,V} \otimes \mathrm{id}_W)\\ = (\mathrm{id}_{({}^{U\,V}W)} \otimes c_{U,V})\,(c_{U,{}^{V}W} \otimes \mathrm{id}_V)\,(\mathrm{id}_U \otimes c_{V,W}).\end{aligned} \tag{2.2.e}$$

Indeed, by (2.2.d) and (2.2.3),

$$(c_{{}^{U}V,{}^{U}W} \otimes \mathrm{id}_U)\,(\mathrm{id}_{({}^{U}V)} \otimes c_{U,W})\,(c_{U,V} \otimes \mathrm{id}_W) = ({}^{U}(c_{V,W}) \otimes \mathrm{id}_U)\, c_{U,V\otimes W}.$$

Applying (2.2.b) to $f = \mathrm{id}_U$, $g = c_{V,W}$ and using (2.2.d), we obtain

$$\begin{aligned}({}^U(c_{V,W}) \otimes \mathrm{id}_U)\, c_{U,V\otimes W} &= c_{U,{}^V W\otimes V}(\mathrm{id}_U \otimes c_{V,W}) \\ &= (\mathrm{id}_{({}^U\,{}^V W)} \otimes c_{U,V})\,(c_{U,{}^V W} \otimes \mathrm{id}_V)\,(\mathrm{id}_U \otimes c_{V,W}).\end{aligned}$$

If $\mathcal{C}$ is strict, then applying (2.2.c) and (2.2.d) to $U = V = \mathbb{1}$ and $V = W = \mathbb{1}$, respectively, and using the invertibility of $c_{U,\mathbb{1}}$, $c_{\mathbb{1},U}$, we obtain

$$c_{U,\mathbb{1}} = c_{\mathbb{1},U} = \mathrm{id}_U \tag{2.2.f}$$

for any object $U \in \mathcal{C}$.

2.3 Twist in G-categories. A *twist* in a braided (crossed) G-category $\mathcal{C}$ is a family of invertible morphisms $\{\theta_U : U \to {}^U U\}_{U\in\mathcal{C}}$ satisfying the following conditions:

(2.3.1) for any morphism $f : U \to V$ in $\mathcal{C}$ with U, V lying in the same component of $\mathcal{C}$, we have $\theta_V f = ({}^U f)\,\theta_U$;

(2.3.2) for any $U \in \mathcal{C}$,

$$(\theta_U \otimes \mathrm{id}_{U^*}) b_U = (\mathrm{id}_{({}^U U)} \otimes \theta_{({}^U U)^*}) b_{({}^U U)};$$

(2.3.3) for any objects $U, V \in \mathcal{C}$,

$$\theta_{U\otimes V} = c_{({}^U V)V,{}^U U}\, c_{U,{}^U V}\,(\theta_U \otimes \theta_V); \tag{2.3.a}$$

(2.3.4) the action of G on $\mathcal{C}$ preserves the twist, i.e., for any $\alpha \in G$ and any $V \in \mathcal{C}$, we have $\varphi_\alpha(\theta_V) = \theta_{\varphi_\alpha(V)}$.

As an exercise, the reader may check that the morphisms on both sides of the equations (2.3.1)–(2.3.4) have the same source and target. If $\mathcal{C}$ is strict, then it follows from (2.2.f) and (2.3.3) that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

A braided crossed G-category endowed with a twist is called a *ribbon crossed G-category*. For $G = 1$, we obtain the standard definition of a ribbon monoidal category.

The neutral component $\mathcal{C}_1$ of a ribbon crossed G-category $\mathcal{C}$ is a ribbon category in the usual sense of the word. Note also that every ribbon crossed G-category is equivalent to a strict ribbon crossed G-category in a canonical way.

2.4 Dual morphisms. Condition (2.3.2) is better understood when it is rewritten in terms of dual morphisms. For a morphism $f : U \to V$ in a monoidal category with left duality, the *dual* (or *transpose*) morphism $f^* : V^* \to U^*$ is defined by

$$f^* = r^{-1}_{U^*}(d_V \otimes \mathrm{id}_{U^*})\, a^{-1}_{V^*,V,U^*}(\mathrm{id}_{V^*} \otimes (f \otimes \mathrm{id}_{U^*}))\,(\mathrm{id}_{V^*} \otimes b_U)\, l_{V^*}.$$

It follows from (1.1.g) that $(\mathrm{id}_U)^* = \mathrm{id}_{U^*}$. It is well known (and can be easily deduced from the definitions) that $(fg)^* = g^* f^*$ for any composable morphisms f and g. Condition (2.3.2) can be shown to be equivalent to

$$(\theta_U)^* = \theta_{U(U^*)}. \tag{2.4.a}$$

2.5 Operations on ribbon group-categories. The operations on group-categories defined in Section 1.4 can be adapted to the setting of crossed (resp. braided, ribbon) group-categories. Given a group homomorphism $q: G' \to G$, any crossed G-category $\mathcal{C}$ yields a crossed G'-category $\mathcal{C}' = q^*(\mathcal{C})$ with action of G' defined by $\varphi_\alpha = \varphi_{q(\alpha)}: \mathcal{C}'_\beta \to \mathcal{C}'_{\alpha\beta\alpha^{-1}}$ for $\alpha, \beta \in G'$. A braiding (resp. twist) in $\mathcal{C}$ induces a braiding (resp. twist) in $\mathcal{C}'$ in the obvious way. In particular, if $G' \subset G$ is a subgroup of G, then any crossed (resp. braided, ribbon) G-category $\mathcal{C} = \coprod_{\alpha\in G} \mathcal{C}_\alpha$ induces a crossed (resp. braided, ribbon) G'-category $\coprod_{\alpha\in G'} \mathcal{C}_\alpha$.

A crossed (resp. braided, ribbon) group-category can be pushed forward along group epimorphisms whose kernels act trivially on the category. Consider a group epimorphism $q: G' \to G$ whose kernel acts as the identity on a crossed (resp. braided, ribbon) G'-category $\mathcal{C}'$. Then the action of G' on $\mathcal{C}'$ induces an action of G on the push-forward G-category $q_*(\mathcal{C}')$ defined in Section 1.4. A braiding (resp. twist) in $\mathcal{C}'$ induces a braiding (resp. twist) in $q_*(\mathcal{C}')$ in the obvious way.

Given a family of crossed G-categories $\{\mathcal{C}^i\}_{i\in I}$, the direct product $\times_i \mathcal{C}^i$ is a crossed G-category. The action of $\alpha \in G$ on objects and morphisms is defined by

$$\varphi_\alpha(\{U_i\}_{i\in I}) = \{\varphi_\alpha(U_i)\}_{i\in I}, \quad \varphi_\alpha(\{f_i\}_{i\in I}) = \{\varphi_\alpha(f_i)\}_{i\in I}. \tag{2.5.a}$$

If $\{\mathcal{C}^i\}_{i\in I}$ are braided (resp. ribbon) G-categories then $\times_i \mathcal{C}^i$ is a braided (resp. ribbon) G-category: braiding and twist are defined coordinate-wise and their coordinates are the given braidings and twists in $\{\mathcal{C}^i\}_{i\in I}$, respectively.

For a finite family of crossed G-categories $\{\mathcal{C}^i\}_{i\in I}$, the tensor product $\bigotimes_i \mathcal{C}^i$ is a crossed G-category. The action of $\alpha \in G$ on objects is defined by (2.5.a). The action of $\alpha \in G$ on morphisms is defined on the generators by $\varphi_\alpha(\otimes_i f_i) = \bigotimes_i \varphi_\alpha(f_i)$ and extends to arbitrary morphisms by K-linearity. If $\{\mathcal{C}^i\}_{i\in I}$ are braided (resp. ribbon) G-categories then $\otimes_i \mathcal{C}^i$ is a braided (resp. ribbon) G-category: braiding and twist are obtained from the corresponding morphisms in $\times_i \mathcal{C}^i$ via the canonical functor $\times_i \mathcal{C}^i \to \bigotimes_i \mathcal{C}^i$.

We define an involutive transformation of crossed G-categories called *reflection*. Let $\mathcal{C} = \coprod_{\alpha\in G} \mathcal{C}_\alpha$ be a crossed G-category with tensor product $\otimes$, duality $*$, structural morphisms a, l, r, b, d and G-action $\varphi: G \to \mathrm{Aut}(\mathcal{C})$. We define a crossed G-category $\overline{\mathcal{C}} = \coprod_{\alpha\in G} \overline{\mathcal{C}}_\alpha$ with tensor product $\bar\otimes$, duality $\star$, structural morphisms $\bar b$, $\bar l$, $\bar r$, $\bar b$, $\bar d$ and G-action $\bar\varphi: G \to \mathrm{Aut}(\overline{\mathcal{C}})$ as follows:

- $\overline{\mathcal{C}} = \mathcal{C}$ as categories (but not as monoidal categories);
- $\overline{\mathcal{C}}_\alpha = \mathcal{C}_{\alpha^{-1}}$ as categories for all $\alpha \in G$;
- $\mathbb{1}_{\overline{\mathcal{C}}} = \mathbb{1}_{\mathcal{C}}$;
- for any objects $U \in \overline{\mathcal{C}}_\alpha, V \in \overline{\mathcal{C}}_\beta$, set $U \mathbin{\bar\otimes} V = \varphi_{\beta^{-1}}(U) \otimes V \in \overline{\mathcal{C}}_{\alpha\beta}$;
- for any morphisms $f: U \to U'$ and $g: V \to V'$ in $\overline{\mathcal{C}}$, where $U \in \overline{\mathcal{C}}_\alpha$,

$U' \in \overline{\mathcal{C}}_{\alpha'}$, $V \in \overline{\mathcal{C}}_\beta$, $V' \in \overline{\mathcal{C}}_{\beta'}$, set

$$f \mathbin{\bar{\otimes}} g = \begin{cases} \varphi_{\beta^{-1}}(f) \otimes g \in \mathrm{Hom}_{\overline{\mathcal{C}}}(U \mathbin{\bar{\otimes}} V, U' \mathbin{\bar{\otimes}} V') & \text{if } \beta = \beta', \\ 0 \in \mathrm{Hom}_{\overline{\mathcal{C}}}(U \mathbin{\bar{\otimes}} V, U' \mathbin{\bar{\otimes}} V') & \text{if } \beta \neq \beta'; \end{cases}$$

- for $U \in \overline{\mathcal{C}}_\alpha$, set $U^\star = \varphi_\alpha(U^*) \in \overline{\mathcal{C}}_{\alpha^{-1}}$ and

$$\bar{l}_U = l_U, \quad \bar{r}_U = r_U, \quad \bar{d}_U = d_U, \quad \bar{b}_U = \varphi_\alpha(b_U);$$

- for objects $U \in \overline{\mathcal{C}}_\alpha$, $V \in \overline{\mathcal{C}}_\beta$, $W \in \overline{\mathcal{C}}_\gamma$, set

$$\bar{b}_{U,V,W} = a_{\varphi_{\gamma^{-1}\beta^{-1}}(U),\varphi_{\gamma^{-1}}(V),W};$$

- for $\alpha \in G$, set $\bar{\varphi}_\alpha = \varphi_\alpha$.

A routine check shows that $\overline{\mathcal{C}}$ is a crossed G-category. Moreover, if c and θ are a braiding and a twist in $\mathcal{C}$, respectively, then the formulas

$$\bar{c}_{U,V} = c^{-1}_{V,U} \quad \text{and} \quad \bar{\theta}_U = \theta^{-1}_{\varphi_\alpha(U)} \quad \text{for } U \in \overline{\mathcal{C}}_\alpha,$$

define a braiding and a twist in $\overline{\mathcal{C}}$. We call $\overline{\mathcal{C}}$ the *mirror image* of $\mathcal{C}$. Its neutral component $\overline{\mathcal{C}}_1$ is the mirror image of $\mathcal{C}_1$ in the sense of [Tu2], Section I.1.4. It is easy to see that $\overline{\overline{\mathcal{C}}} = \mathcal{C}$.

2.6 Example. Consider the G-category $\mathcal{C}$ defined in Section 1.3 and assume that both a and b are invariant under conjugation, i.e.,

$$a_{\delta\alpha\delta^{-1},\delta\beta\delta^{-1},\delta\gamma\delta^{-1}} = a_{\alpha,\beta,\gamma}, \tag{2.6.a}$$

and $b_{\delta\alpha\delta^{-1}} = b_\alpha$ for any $\alpha, \beta, \gamma, \delta \in G$. Then $\mathcal{C}$ is a crossed G-category as follows. For $\alpha, \beta \in G$, set $\varphi_\alpha(V_\beta) = V_{\alpha\beta\alpha^{-1}}$. This extends to morphisms in $\mathcal{C}$ in the obvious way since all non-zero morphisms in $\mathcal{C}$ are proportional to the identity endomorphisms of objects. The resulting functor $\varphi_\alpha : \mathcal{C} \to \mathcal{C}$ preserves all the structural morphisms in $\mathcal{C}$ since a and b are conjugation invariant. To construct specific examples we can take $b = 1$. Finding conjugation invariant 3-cocycles is a delicate task. Obvious examples: the trivial cocycle $a = 1$; any 3-cocycle in the case of abelian G.

A braiding in $\mathcal{C}$ is given by a family $\{c_{\alpha,\beta} \in K^*\}_{\alpha,\beta\in G}$ where $c_{\alpha,\beta}$ determines the braiding morphism

$$V_{\alpha\beta} = V_\alpha \otimes V_\beta \to \varphi_\alpha(V_\beta) \otimes V_\alpha = V_{\alpha\beta\alpha^{-1}} \otimes V_\alpha = V_{\alpha\beta}. \tag{2.6.b}$$

The conditions on the braiding can be reformulated as the following identities: for all $\alpha, \beta, \gamma, \delta \in G$,

$$c_{\delta\alpha\delta^{-1},\delta\beta\delta^{-1}} = c_{\alpha,\beta}, \tag{2.6.c}$$

$$c_{\alpha\beta,\gamma} = c_{\beta,\gamma}\, c_{\alpha,\beta\gamma\beta^{-1}}\, a_{\alpha,\beta,\gamma}\, a^{-1}_{\alpha,\beta\gamma\beta^{-1},\beta}\, a_{\alpha\beta\gamma\beta^{-1}\alpha^{-1},\alpha,\beta}, \tag{2.6.d}$$

$$c_{\alpha,\beta\gamma} = c_{\alpha,\beta}\, c_{\alpha,\gamma}\, a^{-1}_{\alpha,\beta,\gamma}\, a_{\alpha\beta\alpha^{-1},\alpha,\gamma}\, a^{-1}_{\beta,\gamma,\alpha}. \tag{2.6.e}$$

The identity (2.6.d) can be rewritten in a more convenient form using (2.6.c). Namely, observe that $c_{\alpha,\beta\gamma\beta^{-1}} = c_{\delta,\gamma}$ with $\delta = \beta^{-1}\alpha\beta$. Now, $\alpha\beta = \beta\delta$ which gives the following equivalent form of (2.6.d):

$$c_{\beta\delta,\gamma} = c_{\beta,\gamma}\, c_{\delta,\gamma}\, a_{\beta\delta\beta^{-1},\beta,\gamma}\, a^{-1}_{\delta,\gamma,\beta}\, a_{\delta\gamma\delta^{-1},\delta,\beta}. \tag{2.6.f}$$

A direct computation shows that (2.6.c) follows from (1.3.a) and (2.6.a), (2.6.e) , (2.6.f).

The definition of a twist in $\mathcal{C}$ simplifies considerably since ${}^{U}U = U$ for all $U \in \mathcal{C}$ (this is not generally required for a twist in a G-category). Given a braiding $\{c_{\alpha,\beta} \in K^*\}_{\alpha,\beta\in G}$ in $\mathcal{C}$, a twist in $\mathcal{C}$ is determined by a family $\{\theta_\alpha \in K^*\}_{\alpha\in G}$ (where θ_α is the twist $V_\alpha \to V_\alpha = \varphi_\alpha(V_\alpha)$) such that for all $\alpha, \beta \in G$,

$$\theta_{\alpha\beta} = c_{\alpha,\beta} c_{\beta,\alpha} \theta_\alpha \theta_\beta, \tag{2.6.g}$$

$$\theta_{\alpha^{-1}} = \theta_\alpha. \tag{2.6.h}$$

Formula (2.6.g) implies that $\theta_{\alpha\beta} = \theta_{\beta\alpha}$ so that θ is conjugation invariant.

To sum up, a conjugation invariant tuple (a, b, c, θ) satisfying (1.3.a), (2.6.e)–(2.6.h) gives rise to a ribbon crossed G-category $\mathcal{C} = \mathcal{C}(a, b, c, \theta)$. Such tuples (a, b, c, θ) form a group under pointwise multiplication. This group operation corresponds to tensor multiplication of the G-categories $\mathcal{C}(a, b, c, \theta)$. The reflection of ribbon G-categories defined in Section 2.5 corresponds to the following involution in the set of tuples (a, b, c, θ):

$$\bar{b}_{\alpha,\beta,\gamma} = a_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1},\gamma^{-1}}, \quad \bar{b}_\alpha = b_{\alpha^{-1}}, \quad \bar{c}_{\alpha,\beta} = c^{-1}_{\beta^{-1},\alpha^{-1}}, \quad \bar{\theta}_\alpha = \theta_\alpha^{-1}, \tag{2.6.i}$$

for $\alpha, \beta, \gamma \in G$. We shall further discuss equations (2.6.e)–(2.6.h) in Section VIII.4.

If $H \subset G$ is a subgroup of the center of G then the action of H on $\mathcal{C}(a, b, c, \theta)$ is trivial so that we can push $\mathcal{C}(a, b, c, \theta)$ forward along the projection $G \to G/H$. This gives a ribbon crossed (G/H)-category.

2.7 Remarks. 1. The objects of a crossed G-category $(\mathcal{C}, \varphi\colon G \to \mathrm{Aut}(\mathcal{C}))$ form a G-automorphic set in terminology of Brieskorn [Br] or a G-rack in terminology of Fenn and Rourke [FR2]. Recall that a G-*rack* is a set X equipped with a left action of G and a map $\partial\colon X \to G$ such that $\partial(\alpha a) = \alpha\partial(a)\alpha^{-1}$ for all $\alpha \in G$, $a \in X$. The underlying G-rack of $\mathcal{C}$ comprises the set of objects of $\mathcal{C}$, the action of G on this set induced by φ and the map assigning $\alpha \in G$ to any object of $\mathcal{C}_\alpha$.

2. A braiding in a crossed G-category in general is not a braiding in the underlying monoidal category in the usual sense of the word. There is one exceptional case. Namely, assume that G is abelian and $\mathcal{C}$ is a G-category as in Section 1.2. The trivial homomorphism $\varphi = 1\colon G \to \mathrm{Aut}(\mathcal{C})$ turns $\mathcal{C}$ into a crossed G-category. It is clear that a braiding (resp. twist) in this crossed G-category is a braiding (resp. twist) in $\mathcal{C}$ in the usual sense of the word.

VI.3 Colored G-tangles and their invariants

3.1 Colored G-links. It is well known that a framed oriented link in S^3 whose components are colored with objects of a ribbon category gives rise to an invariant lying in the ground ring of this category; see [Tu2]. To adapt this theory to the present setting, we introduce G-links and their colorings. For the sake of future references, we consider links in an arbitrary connected oriented 3-manifold W.

Let $\ell = \ell_1 \cup \cdots \cup \ell_n \subset W$ be an oriented n-component link in W with $n \geq 0$. Denote the 3-manifold $W - \ell$ by C_ℓ where C stands for complement. We say that ℓ is *framed* if each of its component ℓ_i is provided with a *longitude* $\tilde{\ell}_i \subset C_\ell$ which goes very closely along ℓ_i (it may wind around ℓ_i several times). Set $\tilde{\ell} = \bigcup_{i=1}^{n} \tilde{\ell}_i$. For a path $\gamma : [0,1] \to C_\ell$ connecting a point $z = \gamma(0)$ to a point $\gamma(1) \in \tilde{\ell}_i$, denote by $\mu_\gamma \in \pi_1(C_\ell, z)$ the (homotopy) meridian of ℓ_i represented by the loop $\gamma m_i \gamma^{-1}$, where m_i is a small loop in C_ℓ encircling ℓ_i with linking number $+1$. We similarly define a (homotopy) longitude $\lambda_\gamma = [\gamma \tilde{\ell}_i \gamma^{-1}] \in \pi_1(C_\ell, z)$, where the square brackets denote the homotopy class of a loop and the circle $\tilde{\ell}_i$ is viewed as a loop beginning and ending in $\gamma(1)$. Both μ_γ and λ_γ are invariant under homotopies of γ fixing $\gamma(0) = z$ and keeping $\gamma(1)$ on $\tilde{\ell}$. Clearly, μ_γ and λ_γ commute in $\pi_1(C_\ell, z)$. If β is a loop in (C_ℓ, z) (i.e., a path in C_ℓ beginning and ending in z), then

$$\mu_{\beta\gamma} = [\beta]\, \mu_\gamma\, [\beta]^{-1} \quad \text{and} \quad \lambda_{\beta\gamma} = [\beta]\, \lambda_\gamma\, [\beta]^{-1}.$$

For a group G, by a G-*link* in W, we shall mean a triple (a framed oriented link $\ell \subset W$, a base point $z \in C_\ell$, a group homomorphism $g : \pi_1(C_\ell, z) \to G$).

Fix a crossed G-category $(\mathcal{C}, \varphi : G \to \mathrm{Aut}(\mathcal{C}))$ which we call the *category of colors*. A $\mathcal{C}$-*coloring* or, shorter, a *coloring* of a G-link (ℓ, z, g) is a function u which assigns to every path $\gamma : [0,1] \to C_\ell$ with $\gamma(0) = z$ and $\gamma(1) \in \tilde{\ell}$ an object $u_\gamma \in \mathcal{C}_{g(\mu_\gamma)}$ such that the following two conditions are met:

(i) u_γ is preserved under homotopies of γ fixing $\gamma(0) = z$ and keeping $\gamma(1)$ on $\tilde{\ell}$;

(ii) if β is a loop in (C_ℓ, z), then $u_{\beta\gamma} = \varphi_{g([\beta])}(u_\gamma)$.

Pushing the endpoint $\gamma(1) \in \tilde{\ell}$ of a path γ as above along the corresponding component of $\tilde{\ell}$ we can deform γ into a path homotopic to $\lambda_\gamma \gamma$. Conditions (i) and (ii) imply that

$$\varphi_{g(\lambda_\gamma)}(u_\gamma) = u_\gamma. \tag{3.1.a}$$

We shall see below (in Lemma 3.2.1) that a coloring of an n-component G-link $(\ell = \ell_1 \cup \cdots \cup \ell_n, z, g)$ is uniquely determined by the objects of $\mathcal{C}$ associated to any given n paths $\gamma_1, \ldots, \gamma_n$ connecting z to $\tilde{\ell}_1, \ldots, \tilde{\ell}_n$, respectively. In the role of these objects we can take any objects of $\mathcal{C}_{g(\mu_{\gamma_1})}, \ldots, \mathcal{C}_{g(\mu_{\gamma_n})}$ satisfying (3.1.a).

A G-link endowed with a $\mathcal{C}$-coloring is said to be $\mathcal{C}$-*colored* or briefly *colored*. The notion of an ambient isotopy in W applies to G-links and colored G-links in W in the obvious way. This allows us to consider the (ambient) isotopy classes of such links.

The structure of a colored G-link (ℓ, z, g, u) in W can be transferred along paths in $C_\ell = W - \ell$ relating various base points. More precisely, let $\rho\colon [0,1] \to C_\ell$ be a path with $\rho(0) = z$. We define a new colored G-link

$$(\ell' = \ell, z', g'\colon \pi_1(C_\ell, z') \to G, u')$$

by $z' = \rho(1)$, $g'([\alpha]) = g([\rho\alpha\rho^{-1}])$ for any loop α in (C_ℓ, z'), and $u'_\gamma = u_{\rho\gamma}$ for any path γ in C_ℓ leading from z' to $\tilde{\ell}$. It is clear that the transfers along homotopic paths (with the same endpoints) give the same results. The transfer preserves the ambient isotopy class of the colored G-link (ℓ, z, g, u): it is ambient isotopic to (ℓ, z', g', u') via an isotopy of W in itself pushing z along ρ and fixing a neighborhood of ℓ pointwise.

Although we shall not need it, note that the notion of a G-link can be reformulated in terms of a principal G-bundle over the link complement. A $\mathcal{C}$-colored G-link can be defined in terms of G-racks (cf. Remark 2.7.1) as a framed oriented link endowed with a homomorphism of its fundamental rack, defined in [FR2], into the underlying G-rack of $\mathcal{C}$.

3.2 Colored G-tangles. A tangle consists of a finite number of disjoint oriented circles and segments embedded in $\mathbb{R}^2 \times [0,1]$. More formally, by a *tangle* with $k \geq 0$ inputs and $l \geq 0$ outputs we mean a compact oriented 1-dimensional manifold $T \subset \mathbb{R}^2 \times [0,1]$ with bottom endpoints (inputs) $(r, 0, 0)$, $r = 1, \ldots, k$ and top endpoints (outputs) $(s, 0, 1)$, $s = 1, \ldots, l$. It is understood that T meets the planes $\mathbb{R}^2 \times 0$ and $\mathbb{R}^2 \times 1$ only at ∂T and is orthogonal to these planes at ∂T. Set

$$C_T = (\mathbb{R}^2 \times [0,1]) - T.$$

We say that T is *framed* if each its component t is provided with a longitude $\tilde{t} \subset C_T$ which goes very closely along t. Clearly, $\tilde{t}$ is a segment (resp. a circle) if t is a segment (resp. a circle). We always assume that the longitudes of the segment components of T have the endpoints $(r, -\delta, 0)$, $r = 1, \ldots, k$ and $(s, -\delta, 1)$, $s = 1, \ldots, l$ with small $\delta > 0$. Set $\widetilde{T} = \bigcup_t \tilde{t}$, where t runs over all components of T.

As the base point z of C_T we choose a point with negative second coordinate $z_2 \ll 0$ such that $T \subset \mathbb{R} \times [z_2 + 1, \infty) \times [0,1]$. The set of such z is contractible. This allows us to suppress the base point from the notation for $\pi_1(C_T)$.

For each path $\gamma\colon [0,1] \to C_T$ connecting the base point of C_T to $\widetilde{T}$, we introduce a meridian $\mu_\gamma \in \pi_1(C_T)$ as in Section 3.1. If $\gamma(1)$ lies on a circle component of $\widetilde{T}$, then we also have a longitude $\lambda_\gamma \in \pi_1(C_T)$. Both μ_γ and λ_γ are invariant under homotopies of γ fixing $\gamma(0)$ and keeping $\gamma(1)$ on $\widetilde{T}$. Clearly, μ_γ and λ_γ commute in $\pi_1(C_T)$.

For a group G, by a *G-tangle* we mean a pair (a framed tangle T, a group homomorphism $g\colon \pi_1(C_T) \to G$). The definition of a $\mathcal{C}$-coloring of a G-link extends to G-tangles word for word. A G-tangle endowed with a $\mathcal{C}$-coloring is said to be *$\mathcal{C}$-colored* or, shorter, *colored*.

As usual, we shall consider G-tangles and colored G-tangles up to ambient isotopy in $\mathbb{R}^2 \times [0,1]$ constant on the endpoints. The G-tangles (resp. colored G-tangles) with 0 inputs and 0 outputs are nothing but G-links (resp. colored G-links) in $\mathbb{R}^2 \times (0,1)$.

Let $T = (T, g, u)$ be a colored G-tangle. With the r-th input $(r, 0, 0)$ of T we associate a triple $(\varepsilon_r = \pm, \alpha_r \in G, U_r \in \mathcal{C}_{\alpha_r})$ as follows. Set $\varepsilon_r = +$ if the segment component of T incident to the r-th input is directed out of $\mathbb{R}^2 \times [0,1]$ and set $\varepsilon_r = -$ otherwise. Let γ_r be the path in C_T leading from the base point $z = (z_1, z_2, z_3)$ of C_T to the r-th input $(r, -\delta, 0)$ of $\widetilde{T}$ and defined as composition of the linear paths from z to $(r, z_2, 0)$ and from $(r, z_2, 0)$ to $(r, -\delta, 0)$. We call γ_r the *canonical path* associated to the r-th input of T. The meridian $\mu_r = \mu_{\gamma_r} \in \pi_1(C_T)$ is called the *canonical meridian* associated to the r-th input of T. Set $\alpha_r = g(\mu_r) \in G$ and $U_r = u_{\gamma_r} \in \mathcal{C}_{\alpha_r}$. The sequence $(\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)$ (where k is the number of inputs of T) is called the *source* of T. Similarly, with the s-th output $(s, 0, 1)$ of T we associate a triple $(\varepsilon^s = \pm, \alpha^s \in G, U^s \in \mathcal{C}_{\alpha^s})$ as follows. Set $\varepsilon^s = +$ if the segment component of T incident to the s-th output is directed inside $\mathbb{R}^2 \times [0,1]$ and set $\varepsilon^s = -$ otherwise. Let γ^s be the canonical path leading from the base point $z = (z_1, z_2, z_3)$ of C_T to the s-th output $(s, -\delta, 1)$ of $\widetilde{T}$. It is defined as the composition of the linear paths from z to $(s, z_2, 1)$ and from $(s, z_2, 1)$ to $(s, -\delta, 1)$. Set $\alpha^s = g(\mu_{\gamma^s}) \in G$ and $U^s = u_{\gamma^s} \in \mathcal{C}_{\alpha^s}$. The sequence $(\varepsilon^1, \alpha^1, U^1), \dots, (\varepsilon^l, \alpha^l, U^l)$ (where l is the number of outputs of T) is called the *target* of T. Note the obvious equality

$$\prod_{r=1}^{k} (\alpha_r)^{\varepsilon_r} = \prod_{s=1}^{l} (\alpha^s)^{\varepsilon^s}.$$

3.2.1 Lemma. *Let $(T = t_1 \cup \dots \cup t_n, g\colon \pi_1(C_T) \to G)$ be an n-component G-tangle. Let ρ_i be a path in C_T leading from the base point $z \in C_T$ to a point of $\tilde{t}_i$, where $i = 1, \dots, n$. Let $U_i \in \mathcal{C}_{g(\mu_{\rho_i})}$ for $i = 1, \dots, n$. Assume that for each circle component t_i of T, we have $\varphi_{g(\lambda_{\rho_i})}(U_i) = U_i$. Then there is a unique $\mathcal{C}$-coloring u of T such that $u_{\rho_i} = U_i$ for $i = 1, \dots, n$.*

Proof. Any path $\gamma\colon [0,1] \to C_T$ connecting z to $\tilde{t}_i$ can be deformed fixing $\gamma(0) = z$ and keeping $\gamma(1) \in \tilde{t}_i$ so that $\gamma(1) = \rho_i(1)$. Then

$$u_\gamma = u_{\gamma\rho_i^{-1}\rho_i} = \varphi_{g([\gamma\rho_i^{-1}])}(u_{\rho_i}) = \varphi_{g([\gamma\rho_i^{-1}])}(U_i).$$

This proves the uniqueness of u. Conversely, our assumptions imply that

$$u_\gamma = \varphi_{g([\gamma\rho_i^{-1}])}(U_i) \in \mathcal{C}_{g(\mu_\gamma)} \tag{3.2.a}$$

does not depend on the choice of the deformation of γ used to ensure $\gamma(1) = \rho_i(1)$. It is easy to check that formula (3.2.a) defines a coloring u of T such that $u_{\rho_i} = U_i$ for all $i = 1, \dots, n$. □

An important class of colored G-tangles is formed by colored G-braids. A colored G-tangle $(T, g\colon \pi_1(C_T) \to G, u)$ is a *colored G-braid* if T is a framed oriented braid. Let k be the number of strings of T. Observe that the orientation of

T, the homomorphism g, and the coloring u are uniquely determined by the source $(\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)$ of T. Indeed, the signs $\varepsilon_1, \dots, \varepsilon_k$ determine the orientation of the strings. The group $\pi_1(C_T)$ is free on the canonical meridians $\mu_1, \dots, \mu_k$ corresponding to the inputs of T. Hence, the homomorphism g is determined by $\alpha_1 = g(\mu_1), \dots, \alpha_k = g(\mu_k)$. It follows from the definitions that the coloring u is determined by $U_1, \dots, U_k$. Lemma 3.2.1 implies that, conversely, given a finite sequence $(\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)$ with $\varepsilon_r = \pm, \alpha_r \in G, U_r \in \mathcal{C}_{\alpha_r}$ for $r = 1, \dots, k$, we can enrich (uniquely) any framed braid on k strings to a colored G-braid with source $(\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)$.

3.3 Diagrams for colored G-tangles. It is standard in knot theory to present framed tangles by plane pictures called *tangle diagrams*. A tangle diagram lies in a horizontal strip on the plane of the picture identified with

$$\mathbb{R} \times [0,1] = \mathbb{R} \times 0 \times [0,1] \subset \mathbb{R}^2 \times [0,1].$$

We agree that the first axis is a horizontal line on the page directed to the right, the second axis is orthogonal to the page and is directed from the eye of the reader towards the page, the third axis is a vertical line on the page directed from the bottom to the top. A tangle diagram consists of oriented immersed segments and circles lying in general position with indication of over/undercrossings in all double points. The diagram has the same inputs and outputs as the corresponding tangle. The framing is given by pushing the tangle along the vector $(0, -\delta, 0)$ with small $\delta > 0$. Note that the points with negative second coordinate lie above the page.

It is easy to extend the technique of tangle diagrams to present a G-tangle $(T, g\colon \pi_1(C_T) \to G)$. We first present T by a tangle diagram D. The undercrossings of D split D into disjoint oriented embedded arcs in $\mathbb{R} \times [0,1]$. (We do not break D at the overcrossings). For each of these arcs, say e, consider the linear path in $\mathbb{R}^2 \times [0,1]$ connecting the base point $z \in C_T$ to a point of $e + (0, -\delta, 0)$. In the pictorial language, the point z lies high above D and the path in question is obtained by rushing from z straight to a point lying slightly above e. Denote this path by $\gamma(e)$. We label e with $g_e = g(\mu_{\gamma(e)}) \in G$. In pictures, one usually puts g_e on a small arrow drawn beneath e and crossing e from right to left with respect to the counterclockwise orientation of the plane of the picture. Knowing g_e for all arcs e of D, we can recover the homomorphism g because the meridians $\{\mu_{\gamma(e)}\}_e$ generate $\pi_1(C_T)$. We say that the G-tangle (T, g) is *presented by the diagram* D whose arcs are labeled by elements of G as above. To present a $\mathcal{C}$-colored G-tangle (T, g, u), we additionally endow each arc e of D with the object $u_{\gamma(e)}$ of $\mathcal{C}_{g_e}$. This data uniquely determines (T, g, u).

It is easy to see that a tangle diagram D, whose arcs are labeled by elements of G, presents a G-tangle if and only if the labels satisfy the following local condition:

$(*)$ Encircling a double point of D and multiplying the corresponding four elements of G we always obtain $1 \in G$. (It is understood that crossing an arc e from right to left we read g_e while crossing e from left to right we read g_e^{-1}.)

Consider now a tangle diagram D whose arcs e are labeled with pairs ($g_e \in G, U_e \in \mathcal{C}_{g_e}$). It is easy to see that D presents a colored G-tangle if and only if the labels $\{g_e\}_e$ satisfy $(*)$ and the labels $\{U_e\}_e$ satisfy the following local condition:

$(**)$ For any double point d of D, consider the three arcs e, f, h of D incident to d such that in a neighborhood of d they appear as the overcrossing, the incoming undercrossing, and the outgoing undercrossing, respectively. Then $U_f = (\varphi_{g_e})^\varepsilon(U_h)$, where $\varepsilon = \pm 1$ is the sign of d; see Figures VI.1 and VI.2.

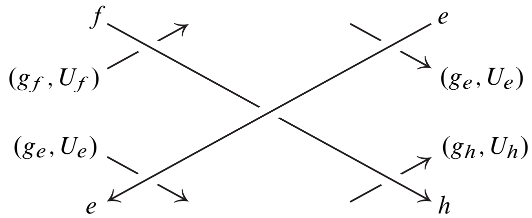


Figure VI.1. At a positive crossing $g_f = g_e g_h g_e^{-1}$ and $U_f = \varphi_{g_e}(U_h)$.

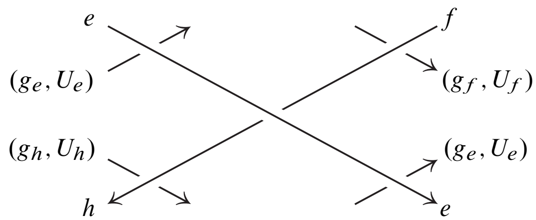


Figure VI.2. At a negative crossing $g_f = g_e^{-1} g_h g_e$ and $U_f = \varphi_{g_e}^{-1}(U_h)$.

The necessity of $(**)$ follows from the definition of a coloring and the obvious equality $\gamma(f) = (\mu_{\gamma(e)})^\varepsilon \gamma(h)$. The sufficiency of $(**)$ follows from Lemma 3.2.1 and the fact that $(**)$ implies (3.1.a) for all circle components. In the sequel, by a diagram of a colored G-tangle we mean its diagram labeled as above so that both conditions $(*)$ and $(**)$ are met.

An example of a diagram of a colored G-tangle with 4 inputs and 2 outputs is given in Figure VI.3. Here α, β, γ are arbitrary elements of G and $\beta' = \gamma\beta\gamma^{-1}$, $\beta'' = \alpha^{-1}\beta'\alpha$. The symbols U, V, W stand for arbitrary objects of the categories $\mathcal{C}_\alpha, \mathcal{C}_\beta, \mathcal{C}_\gamma$, respectively, and

$$V' = \varphi_\gamma(V) \in \mathcal{C}_{\beta'}, \quad V'' = \varphi_\alpha^{-1}(V') \in \mathcal{C}_{\beta''}.$$

The source of this colored G-tangle is the sequence

$$(+, \alpha, U), \quad (-, \beta'', V''), \quad (-, \alpha, U), \quad (+, \gamma, W).$$

The target of this colored G-tangle is the sequence $(+, \gamma, W)$, $(-, \beta, V)$.

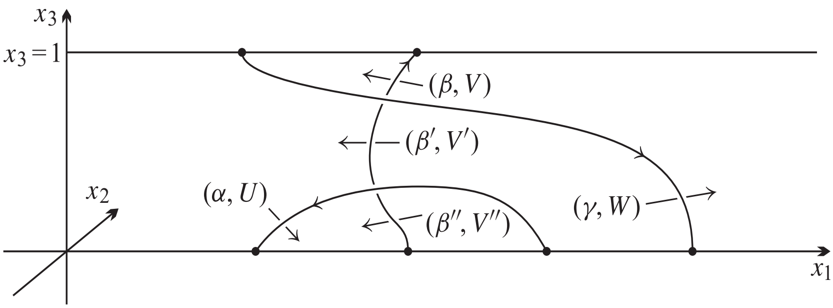


Figure VI.3. A diagram of a colored G-tangle.

The technique of Reidemeister moves on diagrams of framed tangles (see, for instance, [Tu2], Section I.4) extends to the diagrams of colored G-tangles. The key point is that any Reidemeister move on the underlying unlabeled diagram extends uniquely to a move on the labels keeping all the labels outside of the 2-disc where the diagram is modified. As in the standard theory, two diagrams of colored G-tangles represent isotopic colored G-tangles if and only if they can be related by a finite sequence of (labeled) Reidemeister moves.

3.4 Category of colored G-tangles. We define a category of colored G-tangles $\mathcal{T} = \mathcal{T}(G, \mathcal{C}, K)$ as follows. The objects of $\mathcal{T}$ are finite sequences $\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^k$ where $k \geq 0$, $\varepsilon_r = \pm$, $\alpha_r \in G$, $U_r \in \mathcal{C}_{\alpha_r}$ for $r = 1, \dots, k$. A morphism of such sequences $\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^k \to \{(\varepsilon^s, \alpha^s, U^s)\}_{s=1}^l$ in $\mathcal{T}$ is a finite formal sum $\sum_i a_i T_i$, where $a_i \in K$ and T_i is a colored G-tangle with k inputs and l outputs such that $\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^k$ is the source of T_i and $\{(\varepsilon^s, \alpha^s, U^s)\}_{s=1}^l$ is the target of T_i. If the sum $\sum a_i T_i$ consists of only one term $1\,T$, then we say that the corresponding morphism in $\mathcal{T}$ is represented by T. The composition of morphisms $T \circ T'$ represented by colored G-tangles T and T' is obtained by gluing T on the top of T'; this extends by K-linearity to all morphisms. The identity morphism of an object $\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^k$ is represented by the trivial colored G-braid with constant framing and source (and target) $\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^k$. This completes the definition of $\mathcal{T}$.

3.4.1 Lemma. *The category $\mathcal{T}$ is a strict ribbon crossed G-category.*

Proof. The tensor product for the objects of $\mathcal{T}$ is the juxtaposition of sequences. The unit object is the empty sequence. The tensor product of the morphisms in $\mathcal{T}$ represented by colored G-tangles T and T' is obtained by placing a diagram of T' on the right of a diagram of T. The union of these two diagrams represents the colored G-tangle $T \otimes T'$. This extends to arbitrary morphisms in $\mathcal{T}$ by linearity. The associativity morphisms and the structural morphisms l, r in $\mathcal{T}$ are the identities. In this way, $\mathcal{T}$ becomes a strict monoidal category.

The dual of an object

$$U = ((\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)) \in \mathcal{T}$$

is the object

$$U^* = ((-\varepsilon_k, \alpha_k, U_k), (-\varepsilon_{k-1}, \alpha_{k-1}, U_{k-1}), \dots, (-\varepsilon_1, \alpha_1, U_1)).$$

The duality morphisms $b_U : \mathbb{1} \to U \otimes U^*$ and $d_U : U^* \otimes U \to \mathbb{1}$ are defined in the same way as in the usual theory where $G = 1$. They are represented by tangle diagrams consisting of k disjoint concentric cups (resp. caps); see Figure VI.4. The orientations and labels on these cups (resp. caps) are uniquely determined by the data at the outputs (resp. inputs). Formulas (1.1.h) are straightforward.

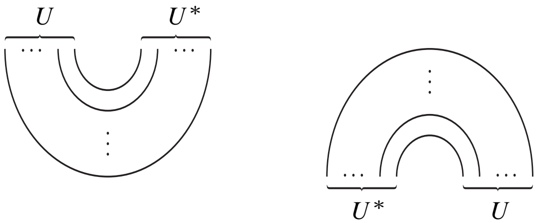


Figure VI.4. The duality morphisms b_U and d_U.

Given $\alpha \in G$, consider a full subcategory $\mathcal{T}_\alpha$ of $\mathcal{T}$ whose objects are sequences $((\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k))$ such that $\prod_{r=1}^{k} (\alpha_r)^{\varepsilon_r} = \alpha$. It is obvious that $\mathcal{T} = \coprod_{\alpha \in G} \mathcal{T}_\alpha$. It follows from the definitions that $\mathcal{T}$ is a G-category.

For each $\alpha \in G$, we define an automorphism φ_α of $\mathcal{T}$ as follows. The action on the objects is given by

$$\varphi_\alpha(\{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^{k}) = \{(\varepsilon_r, \alpha\alpha_r\alpha^{-1}, \varphi_\alpha(U_r))\}_{r=1}^{k},$$

where on the right-hand side we use the action of α on the category of colors $\mathcal{C}$. The automorphism $\varphi_\alpha : \mathcal{T} \to \mathcal{T}$ transforms a colored G-tangle $(T, g : \pi_1(C_T) \to G, u)$ into the colored G-tangle $(T, \alpha g \alpha^{-1} : \pi_1(C_T) \to G, \varphi_\alpha u)$, where $\varphi_\alpha u$ is the coloring of the G-tangle $(T, \alpha g \alpha^{-1})$ assigning to any path γ in C_T leading from the base point of C_T to $\widetilde{T}$ the object $\varphi_\alpha(u_\gamma) \in \mathcal{C}_{\alpha g(\mu_\gamma)\alpha^{-1}}$. This extends to arbitrary morphisms in $\mathcal{T}$ by linearity and yields an automorphism φ_α of $\mathcal{T}$. In this way $\mathcal{T}$ becomes a crossed G-category.

For objects $U = \{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^{k} \in \mathcal{T}_\alpha$ and $V = \{(\mu_s, \beta_s, V_s)\}_{s=1}^{l} \in \mathcal{T}_\beta$ with $\alpha, \beta \in G$, the braiding $U \otimes V \to {}^{U}V \otimes U$ is represented by the same (framed oriented) braid T on $k + l$ strings as in the standard theory. The braid T is presented by a diagram consisting of two families of parallel linear segments: one family consists of k segments going from k leftmost inputs to k rightmost outputs, the second family consists of l segments going from l rightmost inputs to l leftmost outputs and lying below the segments of the first family; see Figure VI.5. We extend T uniquely to a colored G-braid with source

$$U \otimes V = ((\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k), (\mu_1, \beta_1, V_1), \dots, (\mu_l, \beta_l, V_l)).$$

It follows from the definitions that the target of T is then

$$\begin{aligned}&((\mu_1, \alpha\beta_1\alpha^{-1}, \varphi_\alpha(V_1)), \dots, (\mu_l, \alpha\beta_l\alpha^{-1}, \varphi_\alpha(V_l)), (\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k))\\ &\quad = \varphi_\alpha(V) \otimes U = {}^{U}V \otimes U.\end{aligned}$$

Hence T represents a morphism $c_{U,V}\colon U \otimes V \to {}^{U}V \otimes U$ in $\mathcal{T}$. The axioms of a braiding are straightforward.

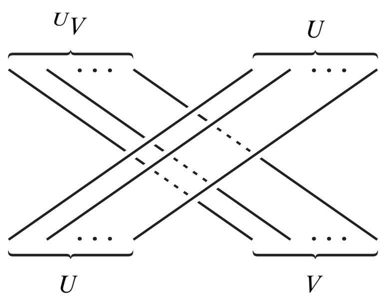


Figure VI.5. The braiding morphism $c_{U,V}$.

For $U = \{(\varepsilon_r, \alpha_r, U_r)\}_{r=1}^{k} \in \mathcal{T}_\alpha$, the twist

$$U \to {}^{U}U = \varphi_\alpha(U) = \{(\varepsilon_r, \alpha\alpha_r\alpha^{-1}, \varphi_\alpha(U_r))\}_{r=1}^{k}$$

is represented by the same framed oriented braid as in the standard theory. It can be obtained from the trivial braid on k strings with constant framing by one full right-hand twist. We extend this oriented framed braid uniquely to a colored G-braid Θ_U with source U. It is easy to deduce from the definitions that the target of Θ_U is ${}^{U}U = \varphi_\alpha(U)$ so that Θ_U represents a morphism $\theta_U\colon U \to {}^{U}U$ in $\mathcal{T}$. The axioms of a twist are straightforward. Thus, $\mathcal{T}$ is a ribbon crossed G-category. □

3.5 Ribbon functors. To formulate our main theorem on colored G-tangles we need a notion of a ribbon G-functor. Consider two crossed G-categories $\mathcal{C}$ and $\mathcal{C}'$. A *monoidal G-functor* from $\mathcal{C}'$ to $\mathcal{C}$ is a covariant functor $F\colon \mathcal{C}' \to \mathcal{C}$ such that

(i) F is K-linear on morphisms;

(ii) $F(\mathbb{1}_{\mathcal{C}'}) = \mathbb{1}_{\mathcal{C}}$ and $F(f \otimes g) = F(f) \otimes F(g)$ for any objects or morphisms f, g in $\mathcal{C}'$;

(iii) F transforms the structural morphisms a, l, r in $\mathcal{C}'$ into the corresponding morphisms in $\mathcal{C}$: for any $U, V, W \in \mathcal{C}'$,

$$F(a_{U,V,W}) = a_{F(U),F(V),F(W)}, \quad F(l_U) = l_{F(U)}, \quad F(r_U) = r_{F(U)};$$

(iv) F maps $\mathcal{C}'_\alpha$ into $\mathcal{C}_\alpha$ for all $\alpha \in G$;

(v) F is equivariant with respect to the action of G, i.e., $F \circ \varphi_\alpha = \varphi_\alpha \circ F$ for any $\alpha \in G$.

Note that we do not require monoidal functors to preserve duality. If $\mathcal{C}$ and $\mathcal{C}'$ are strict, then condition (iii) is superfluous since the structural morphisms in question are the identities.

Assume that $\mathcal{C}, \mathcal{C}'$ are ribbon G-categories. A monoidal G-functor $F : \mathcal{C}' \to \mathcal{C}$ is *ribbon* if it transforms the braiding and twist in $\mathcal{C}'$ into the braiding and twist in $\mathcal{C}$, i.e., for any $U, V \in \mathcal{C}'$,

$$F(c_{U,V}) = c_{F(U),F(V)} \quad \text{and} \quad F(\theta_U) = \theta_{F(U)}.$$

In the next theorem and in the sequel, for an object $U \in \mathcal{C}$ and a sign $\varepsilon = \pm$, we write

$$U^\varepsilon = \begin{cases} U & \text{if } \varepsilon = +, \\ U^* & \text{if } \varepsilon = -. \end{cases}$$

3.6 Theorem. *If $\mathcal{C}$ is a strict ribbon crossed G-category, then there is a unique ribbon monoidal G-functor $F : \mathcal{T} = \mathcal{T}(G, \mathcal{C}, K) \to \mathcal{C}$ such that*

(i) *for any length 1 object (ε, α, U) of $\mathcal{T}$, we have $F((\varepsilon, \alpha, U)) = U^\varepsilon$;*

(ii) *for any $\alpha \in G, U \in \mathcal{C}_\alpha$,*

$$F(b_{(+,\alpha,U)}) = b_U : \mathbb{1}_{\mathcal{C}} \to U \otimes U^* \quad \textit{and} \quad F(d_{(+,\alpha,U)}) = d_U : U^* \otimes U \to \mathbb{1}_{\mathcal{C}}.$$

Theorem 3.6 generalizes the results of [Tu2], Chapter I, where $G = 1$.

The uniqueness in Theorem 3.6 is quite straightforward. The assumption that F is monoidal and condition (i) determine F on all objects:

$$F((\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)) = \bigotimes_{r=1}^{k} (U_r)^{\varepsilon_r}.$$

Next we observe that the morphisms of type

$$(c_{(\varepsilon,\alpha,U),(\varepsilon',\alpha',U')})^{\pm 1}, \quad (\theta_{(\varepsilon,\alpha,U)})^{\pm 1}, \quad b_{(+,\alpha,U)}, \quad d_{(+,\alpha,U)} \tag{3.6.a}$$

(where $\varepsilon, \varepsilon' = \pm, \alpha, \alpha' \in G, U \in \mathcal{C}_\alpha, U' \in \mathcal{C}_{\alpha'}$) generate the category $\mathcal{T}$ in the sense that any morphism in $\mathcal{T}$ can be obtained from these generators by taking tensor product and composition. Given the values of F on these generators, we can compute F on any morphism in $\mathcal{T}$. This proves the uniqueness of F. In particular, the values of F on the colored G-tangles $b_{(-,\alpha,U)}, d_{(-,\alpha,U)}$ with $\alpha \in G, U \in \mathcal{C}_\alpha$ can be computed from the following equalities:

$$d_{(-,\alpha,U)} = d_{(+,\alpha,{}^{U}U)}\, c_{(+,\alpha,{}^{U}U),(-,\alpha,U)}\, (\theta_{(+,\alpha,U)} \otimes \mathrm{id}_{(-,\alpha,U)}), \tag{3.6.b}$$

$$b_{(-,\alpha,U)} = (\mathrm{id}_{(-,\alpha,U)} \otimes (\theta_{(+,\alpha,U)})^{-1})\, (c_{(-,\alpha,U),(+,\alpha,{}^{U}U)})^{-1}\, b_{(+,\alpha,U)}. \tag{3.6.c}$$

A colored G-link (ℓ, z, g, u) in $\mathbb{R}^2 \times (0,1)$ represents an endomorphism of the unit object $\mathbb{1}_\mathcal{T}$ in $\mathcal{T}$ (the empty sequence) and is mapped by F into $F(\ell, z, g, u) \in \mathrm{End}_\mathcal{C}(\mathbb{1}_\mathcal{C})$. It is obvious that any colored G-link in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is ambient isotopic to a unique (up to ambient isotopy) colored G-link in $\mathbb{R}^2 \times (0,1)$. This allows us to apply F to colored G-links in S^3. For instance, an empty G-link $\emptyset$ represents the identity endomorphism of $\mathbb{1}_\mathcal{T}$ and therefore $F(\emptyset) = 1 \in \mathrm{End}_\mathcal{C}(\mathbb{1}_\mathcal{C})$.

In the next section we generalize Theorem 3.6 to colored G-graphs and discuss a proof of this generalization.

VI.4 Colored G-graphs and their invariants

The standard theory of colored tangles generalizes to so-called ribbon graphs. Here we introduce a similar generalization of colored G-tangles.

4.1 Colored G-graphs. We recall the definition of a ribbon graph referring to [Tu2] for details. A *coupon* is a rectangle with a distinguished side called the bottom base; the opposite side is called the top base. A *ribbon graph* Ω in $\mathbb{R}^2 \times [0,1]$ with $k \geq 0$ inputs $\{(r,0,0)\}_{r=1}^k$ and $l \geq 0$ outputs $\{(s,0,1)\}_{s=1}^l$ consists of a finite family of segments, circles, and coupons embedded in $\mathbb{R}^2 \times [0,1]$. We call these segments, circles, and coupons the *strata* of Ω. The inputs and outputs of Ω should be among the endpoints of the segments, all the other endpoints of the segments should lie on the bases of the coupons. Otherwise the strata of Ω should be disjoint and lie in $\mathbb{R}^2 \times (0,1)$. At the inputs and outputs, the segments should be orthogonal to the planes $\mathbb{R}^2 \times 0$ and $\mathbb{R}^2 \times 1$. All the strata of Ω should be oriented and framed so that their framings form a continuous nonsingular vector field on Ω transversal to Ω. This vector field is the *framing* of Ω. At the inputs and outputs, the framing should be given by the vector $(0,-\delta,0)$ with small positive δ. On each coupon the framing should be transversal to the coupon and yield together with the orientation of the coupon the right-handed orientation of $\mathbb{R}^2 \times [0,1]$. Slightly pushing Ω along its framing we obtain a disjoint copy $\tilde{\Omega}$ of Ω. Pushing a segment (resp. a circle, a coupon) t of Ω along the framing we obtain a segment (resp. a circle, a coupon) $\tilde{t} \subset \tilde{\Omega}$. We use here a language slightly different but equivalent to the one in [Tu2], where instead of the framings on segments and circles we consider orthogonal 2-dimensional bands and annuli.

Let $\Omega \subset \mathbb{R}^2 \times [0,1]$ be a ribbon graph. We provide its complement

$$C_\Omega = (\mathbb{R}^2 \times [0,1]) - \Omega$$

with the "canonical" base point z with a big negative second coordinate $z_2 \ll 0$. As in Section 3, we can suppress the base point from the notation for $\pi_1(C_\Omega)$.

We define meridians of segments and circles of Ω as in Sections 3.1 and 3.2. With the inputs and outputs of Ω we associate canonical meridians as in Section 3.2. We also define meridians of a coupon $Q \subset \Omega$ as follows, see Figure VI.6. Pick an oriented interval $q \subset Q$ leading from the top base of Q to the bottom base. For a path γ in C_Ω from the base point z to a point of $\widetilde{Q}$, the meridian $\mu_\gamma \in \pi_1(C_\Omega)$ is represented by the loop $\gamma m \gamma^{-1}$, where $m \subset C_\Omega$ is a small loop encircling Q (in a plane transversal to q) and having linking number $+1$ with q.

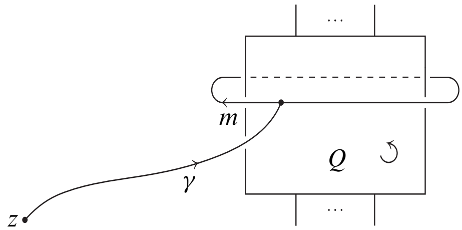


Figure VI.6. A meridian of a coupon Q.

A G-*graph* is a ribbon graph $\Omega \subset \mathbb{R}^2 \times [0,1]$ endowed with a homomorphism $g \colon \pi_1(C_\Omega) \to G$. A G-graph without coupons is just a G-tangle.

Fix a crossed G-category $(\mathcal{C}, \varphi \colon G \to \operatorname{Aut}(\mathcal{C}))$. A *coloring* of a G-graph $(\Omega, g \colon \pi_1(C_\Omega) \to G)$ consists of two functions u and v. The function u assigns to every segment or circle t of Ω and to every path γ in C_Ω from z to a point of $\tilde{t}$ a certain object $u_\gamma \in \mathcal{C}_{g(\mu_\gamma)}$, where $\mu_\gamma \in \pi_1(C_\Omega)$ is the meridian of t determined by γ. This function should satisfy the same conditions as in Section 3, i.e.,

(i) u_γ is preserved under homotopies of γ fixing $\gamma(0)$ and keeping $\gamma(1)$ on $\tilde{t}$;

(ii) if β is a loop in (C_Ω, z), then $u_{\beta\gamma} = \varphi_{g([\beta])}(u_\gamma)$.

The function v assigns to every coupon $Q \subset \Omega$ and to every path γ in C_Ω from z to a point of $\widetilde{Q} \subset \widetilde{\Omega}$ a certain morphism v_γ in $\mathcal{C}_{g(\mu_\gamma)}$ satisfying three conditions:

(i) v_γ is preserved under homotopies of γ fixing $\gamma(0)$ and keeping $\gamma(1)$ on $\widetilde{Q}$;

(ii) if β is a loop in (C_Ω, z), then $v_{\beta\gamma} = \varphi_{g([\beta])}(v_\gamma)$.

To formulate the third condition on v_γ we need some preliminaries. Let $t_1, \dots, t_m$ be the segments of Ω incident to the bottom side of Q and enumerated in the order determined by the direction of this side induced by the orientation of Q. Set $\varepsilon_i = +1$ if t_i is directed out of Q and $\varepsilon_i = -1$ otherwise. Let $t^1, \dots, t^n$ be the segments of Ω incident to the top side of Q and enumerated in the order determined by the direction of this side opposite to the one induced by the orientation of Q. Set $\varepsilon^j = -1$ if t^j is directed out of Q and $\varepsilon^j = +1$ otherwise. For $i = 1, \dots, m$, we can compose the path γ with a path in $\widetilde{Q}$ leading from $\gamma(1) \in \widetilde{Q}$ to the endpoint of $\tilde{t}_i$ lying on the bottom side of $\widetilde{Q}$. The resulting path, γ_i, leads from z to a point of $\tilde{t}_i$. Similarly, for $j = 1, \dots, n$, we can compose γ with a path in $\widetilde{Q}$ leading from $\gamma(1)$ to the endpoint of $\tilde{t}^j$ lying on the top side of $\widetilde{Q}$. The resulting path, γ^j, leads from z to a point of $\tilde{t}^j$. Note that

$$\mu_\gamma = \prod_{i=1}^{m} (\mu_{\gamma_i})^{\varepsilon_i} = \prod_{j=1}^{n} (\mu_{\gamma^j})^{\varepsilon^j} \in \pi_1(C_\Omega).$$

The third condition on v is as follows:

(iii) for any coupon Q of Ω as above and any path γ in C_Ω from z to $\widetilde{Q}$,

$$v_\gamma \in \operatorname{Hom}_{\mathcal{C}}(\otimes_{i=1}^{m} (u_{\gamma_i})^{\varepsilon_i}, \otimes_{j=1}^{n} (u_{\gamma^j})^{\varepsilon^j}),$$

where $u_{\gamma_i} \in \mathcal{C}_{g(\mu_{\gamma_i})}$ and $u_{\gamma^j} \in \mathcal{C}_{g(\mu_{\gamma^j})}$ are the objects determined by u.

A G-graph endowed with a coloring is said to be *colored*. The definition of the source and the target given in Section 3.2 applies to colored G-graphs word for word. The technique of diagrams discussed in Section 3.3 generalizes to colored G-graphs in the obvious way. The coupons should be presented by small rectangles in $\mathbb{R} \times [0, 1]$ with horizontal bottom and top bases such that the bottom base has smaller second coordinate than the top base and the orientation of the coupon is counterclockwise.

We define a category of colored G-graphs $\mathcal{G} = \mathcal{G}(G, \mathcal{C}, K)$ following the lines of Section 3.4. This category has the same objects as $\mathcal{T} = \mathcal{T}(G, \mathcal{C}, K)$. The morphisms in $\mathcal{G}$ are formal linear combinations over K of (the isotopy classes of) colored G-graphs with the given source and target. Composition and tensor product are defined exactly as in $\mathcal{T}$. This turns $\mathcal{G}$ into a strict monoidal category.

4.2 Remark. Sometimes it is convenient to formulate the notions of a G-graph and a colored G-graph using graph exteriors rather than complements. The *exterior* E_Ω of a ribbon graph $\Omega \subset \mathbb{R}^2 \times [0, 1]$ is the complement in $\mathbb{R}^2 \times [0, 1]$ of an open tubular neighborhood of Ω. We choose this neighborhood so that $\widetilde{\Omega} \subset \partial E_\Omega$. It is clear that $E_\Omega \subset C_\Omega$ is a deformation retract of C_Ω so that $\pi_1(E_\Omega) = \pi_1(C_\Omega)$. Repeating the definitions of Section 4.1 but considering only loops and paths in E_Ω we obtain an equivalent definition of a colored G-graph in terms of the exterior.

4.3 Lemma. *The category $\mathcal{G}$ is a strict ribbon crossed G-category containing $\mathcal{T}$ as a monoidal subcategory.*

The proof follows the lines of the proof of Lemma 3.4.1. For $\alpha \in G$, the category $\mathcal{G}_\alpha$ is defined as the full subcategory of $\mathcal{G}$ whose objects are sequences $\{(\varepsilon_r = \pm, \alpha_r \in G, U_r \in \mathcal{C}_{\alpha_r})\}_{r=1}^{k}$ such that $\prod_{r=1}^{k} (\alpha_r)^{\varepsilon_r} = \alpha$. The action of $\alpha \in G$ on $\mathcal{G}$ is obtained by applying $\varphi_\alpha : \mathcal{C} \to \mathcal{C}$ to the objects and morphisms forming the colorings of G-graphs. The braiding, twist, and duality in $\mathcal{G}$ arise from the corresponding morphisms in $\mathcal{T}$ via the obvious inclusion $\mathcal{T} \hookrightarrow \mathcal{G}$.

4.4 Elementary colored G-graphs. Let

$$U = ((\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_m, \alpha_m, U_m)), \quad U' = ((\varepsilon^1, \alpha^1, U^1), \dots, (\varepsilon^n, \alpha^n, U^n))$$

be objects of $\mathcal{G}_\alpha$ with $\alpha \in G$ and $m, n \geq 0$. Let

$$f \in \mathrm{Hom}_{\mathcal{C}}(\otimes_{i=1}^m (U_i)^{\varepsilon_i}, \otimes_{j=1}^n (U^j)^{\varepsilon^j}).$$

With this data we associate a colored G-graph $\Omega = \Omega(U, U', f)$ consisting of one coupon, m vertical intervals attached to the bottom base, and n vertical intervals attached to the top base. The coupon and the intervals lie in $\mathbb{R} \times 0 \times [0,1]$. The framing of Ω is determined everywhere by the vector $(0, -\delta, 0)$ with small $\delta > 0$. The orientation of the bottom (resp. top) intervals is determined by the signs $\varepsilon_1, \dots, \varepsilon_m$ (resp. $\varepsilon^1, \dots, \varepsilon^n$). Note that the group $\pi_1(C_\Omega)$ is generated by the canonical meridians $\mu_1, \dots, \mu_m, \mu^1, \dots, \mu^n \in \pi_1(C_\Omega)$ corresponding to the inputs and outputs of Ω. They are subject to only one relation

$$(\mu_1)^{\varepsilon_1} \dots (\mu_m)^{\varepsilon_m} = (\mu^1)^{\varepsilon^1} \dots (\mu^n)^{\varepsilon^n}. \tag{4.4.a}$$

The formulas $g(\mu_r) = \alpha_r$, $g(\mu^s) = \alpha^s$, where $r = 1, \dots, m$ and $s = 1, \dots, n$ define a group homomorphism $g \colon \pi_1(C_\Omega) \to G$ carrying both sides of (4.4.a) to α. We define $\Omega(U, U', f)$ to be the G-graph (Ω, g) with the coloring (u, v) such that

(i) the source of $\Omega(U, U', f)$ is U and the target of $\Omega(U, U', f)$ is U';

(ii) $v_\gamma = f$ for the linear path γ in C_Ω from the base point of C_Ω to $\widetilde{\Omega}$.

Such a coloring of Ω exists and is unique. We call $\Omega(U, U', f)$ the *elementary colored G-graph* associated with U, U', f.

4.5 Theorem. *If $\mathcal{C}$ is a strict ribbon crossed G-category, then there is a unique ribbon monoidal G-functor $F \colon \mathcal{G} = \mathcal{G}(G, \mathcal{C}, K) \to \mathcal{C}$ extending the functor $\mathcal{T}(G, \mathcal{C}, K) \to \mathcal{C}$ of Theorem 3.6 and such that for any elementary colored G-graph $\Omega(U, U', f)$, we have $F(\Omega(U, U', f)) = f$.*

Theorem 4.5 generalizes Theorem I.2.5 of [Tu2] where $G = 1$.

The uniqueness in Theorem 4.5 is straightforward: it suffices to observe that the morphisms (3.6.a) and the elementary colored G-graphs generate $\mathcal{G}$. The proof of the existence follows the proof of [Tu2], Theorem I.2.5 with appropriate changes. The key role in the proof given in [Tu2] is played by certain commutative diagrams in the target category $\mathcal{C}$. We give here versions of these diagrams in our setting. For any objects $V, W \in \mathcal{C}$ the following two diagrams are commutative:

$$\begin{array}{ccccc}
V = V \otimes \mathbb{1} & \xrightarrow{\mathrm{id}_V \otimes b_W} & V \otimes W \otimes W^* & \xrightarrow{c_{V,W} \otimes \mathrm{id}_{W^*}} & {}^{V}W \otimes V \otimes W^* \\
\downarrow{\scriptstyle c_{V,\mathbb{1}}} & & \downarrow{\scriptstyle c_{V, W \otimes W^*}} & & \downarrow{\scriptstyle \mathrm{id}_{({}^V W)} \otimes c_{V,W^*}} \\
V = \mathbb{1} \otimes V & \xrightarrow{b_{({}^V W)} \otimes \mathrm{id}_V} & {}^{V}W \otimes {}^{V}(W^*) \otimes V & = & {}^{V}W \otimes {}^{V}(W^*) \otimes V,
\end{array}$$

$$\begin{array}{ccccc}
{}^V(W^*)\otimes V\otimes W & \xleftarrow{c_{V,W^*}\otimes \mathrm{id}_W} & V\otimes W^*\otimes W & \xrightarrow{\mathrm{id}_V\otimes d_W} & V\otimes \mathbb{1} = V \\
\Big\downarrow{\scriptstyle \mathrm{id}_{{}^V(W^*)}\otimes c_{V,W}} & & \Big\downarrow{\scriptstyle c_{V,W^*\otimes W}} & & \Big\downarrow{\scriptstyle c_{V,\mathbb{1}}} \\
{}^V(W^*)\otimes {}^VW\otimes V & = & {}^V(W^*)\otimes {}^VW\otimes V & \xrightarrow{d_{({}^VW)}\otimes \mathrm{id}_V} & \mathbb{1}\otimes V = V.
\end{array}$$

(Clearly, $c_{V,\mathbb{1}} = \mathrm{id}_V$.) One should also use similar diagrams with the middle vertical arrow $c_{W^*\otimes W,V}\colon W^*\otimes W\otimes V \to V\otimes W^*\otimes W$ and the diagrams obtained by replacing V, W with W^*, V, respectively. We leave the details to the reader.

4.6 Corollary. *For any object U of a strict ribbon crossed G-category $\mathcal{C}$, there is a canonical isomorphism $\eta_U\colon U \to U^{**} = (U^*)^*$ in $\mathcal{C}$. For any morphism $f\colon U \to V$ in $\mathcal{C}$, the following diagram is commutative:*

$$\begin{array}{ccc}
U & \xrightarrow{\eta_U} & U^{**} \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f^{**}} \\
V & \xrightarrow{\eta_V} & V^{**}.
\end{array}$$

Proof. If $U\in\mathcal{C}_\alpha$ with $\alpha\in G$ then the morphisms

$$\eta_U = (F(d_{(-,\alpha,U)})\otimes \mathrm{id}_{U^{**}})(\mathrm{id}_U\otimes b_{U^*})\colon U \to U^{**}$$

and

$$(d_{U^*}\otimes \mathrm{id}_U)(\mathrm{id}_{U^{**}}\otimes F(b_{(-,\alpha,U)}))\colon U^{**}\to U$$

are mutually inverse isomorphisms. This can be deduced from the definitions using (1.1.h) A pictorial proof uses Theorem 4.5, cf. the proof of Corollary I.2.6.1 in [Tu2]. The second claim is an exercise. □

4.7 Properties of F. We shall now discuss three operations on a colored G-graph (Ω, g, u, v) preserving $F(\Omega, g, u, v)$. Fix a circle stratum ℓ of Ω.

1. The invariant $F(\Omega, g, u, v)$ does not change when the colors of the paths leading to ℓ are replaced by isomorphic colors. More precisely, pick a path γ from the canonical base point of C_Ω to the longitude $\tilde{\ell}$ of ℓ. Let V be an object of $\mathcal{C}_{g(\mu_\gamma)}$ isomorphic to $u_\gamma\in\mathcal{C}_{g(\mu_\gamma)}$. We define a new coloring (u', v) of Ω as follows. For the paths from the base point to the strata of Ω distinct from ℓ, the associated colors are the same as in (u, v). Set $u'_\gamma = V$. This data extends uniquely to a coloring (u', v) of (Ω, g). Then $F(\Omega, g, u, v) = F(\Omega, g, u', v)$. To prove this equality, observe that $F(\Omega, g, u, v)$ does not change when we insert into ℓ two coupons (with one input and one output) whose colors corresponding to γ are mutually inverse isomorphisms $u_\gamma \to V$ and $V \to u_\gamma$. Pushing one of the coupons along ℓ and eventually canceling it with the second coupon, we obtain $F(\Omega, g, u', v)$.

2. The invariant $F(\Omega, g, u, v)$ does not change when we invert the orientation of ℓ and simultaneously replace the colors associated with ℓ by dual objects. The latter

means that for any path γ from the base point of C_Ω to $\tilde{\ell}$, we set $u'_\gamma = (u_\gamma)^*$. For the paths leading to other strata of Ω, the color is preserved. This yields a coloring (u', v) of the G-graph (Ω', g) obtained from (Ω, g) by inverting the orientation of ℓ. Then $F(\Omega, g, u, v) = F(\Omega', g, u', v)$, cf. [Tu2], Corollary I.2.8.1.

3. Let ℓ_1, ℓ_2 be two parallel copies of ℓ determined by the framing and going very closely to ℓ. Consider the ribbon graph $\Omega' = (\Omega - \ell) \cup \ell_1 \cup \ell_2$. To describe the relevant colorings of Ω' we use the language of exteriors. The exterior $E' = E_{\Omega'}$ of Ω' can be obtained from the exterior $E = E_\Omega$ of Ω by attaching the product (a 2-disc with two holes)$\times S^1$ along the 2-torus $S^1 \times S^1 \subset \partial E$ bounding a tubular neighborhood of ℓ. Denote the inclusion homomorphism $\pi_1(E) \to \pi_1(E')$ by h. Assume that there are a homomorphism $g' \colon \pi_1(E') \to G$ and a coloring (u', v') of the G-graph (Ω', g') satisfying the following conditions:

(i) $g = g'h \colon \pi_1(E) \to G$;

(ii) on the paths in E from the base point to the strata of $\Omega - \ell$ the colorings (u, v) and (u', v') coincide;

(iii) if γ is a path in E from the base point to $\tilde{\ell} \subset \partial E$, then composing γ with short paths in $E' - E$ we obtain paths γ_1, γ_2 in E' leading to $\tilde{\ell}_1, \tilde{\ell}_2$, respectively, such that $h(\mu_\gamma) = \mu_{\gamma_1}\mu_{\gamma_2}$ and $u_\gamma = u'_{\gamma_1} \otimes u'_{\gamma_2}$, where $u'_{\gamma_i} \in \mathcal{C}_{g'(\mu_{\gamma_i})}$ for $i = 1, 2$.

We say that the colored G-graph (Ω', g', u', v') is obtained from (Ω, g, u, v) by *doubling* of ℓ. Then $F(\Omega, g, u, v) = F(\Omega', g', u', v')$, cf. [Tu2], Corollary I.2.8.3.

4.8 Remark. One can consider non-associative G-tangles and G-graphs as in the Kontsevich–Vassiliev theory. We shall not pursue this line.

VI.5 Trace, dimension, and algebra of colors

We use Theorem 4.5 to define trace, dimension, and algebra of colors associated with a strict ribbon crossed G-category $\mathcal{C}$.

5.1 Trace. We define the *trace* of an endomorphism $f : U \to U$ of an object $U \in \mathcal{C}$ by

$$\operatorname{tr}(f) = d_{({}^U U)}\, c_{U,U^*}\, (\theta_U f \otimes \operatorname{id}_{U^*})\, b_U \in \operatorname{End}_{\mathcal{C}}(\mathbb{1}) = \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}).$$

It is clear that $\operatorname{tr}(kf) = k \operatorname{tr}(f)$ for any $k \in K$ and $\operatorname{tr}(\varphi_\beta(f)) = \varphi_\beta(\operatorname{tr}(f))$ for any $\beta \in G$, where on the right-hand side φ_β acts on $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$.

If $U \in \mathcal{C}_\alpha$ with $\alpha \in G$, then we can rewrite the definition of $\operatorname{tr}(f : U \to U)$ using (3.6.b) and the functor F from Theorem 4.5:

$$\operatorname{tr}(f) = F(d_{(-,\alpha,U)})\, (f \otimes \operatorname{id}_{U^*})\, b_U \in \operatorname{End}_{\mathcal{C}}(\mathbb{1}).$$

Theorem 4.5 gives a geometric interpretation of $\mathrm{tr}(f)$. Consider the elementary colored G-graph $\Omega_f = \Omega((+,\alpha,U),(+,\alpha,U),f)$ with one input and one output. As in the standard theory we can close Ω_f to obtain a colored G-graph $\widehat{\Omega}_f$ with 0 inputs and 0 outputs. A diagram of $\widehat{\Omega}_f$ is obtained from a (labeled) diagram of Ω_f by connecting the top and bottom endpoints by an arc disjoint from the rest of the diagram. More formally,

$$\widehat{\Omega}_f = d_{(-,\alpha,U)}\,(\Omega_f \otimes \mathrm{id}_{(-,\alpha,U)})\,b_{(+,\alpha,U)}.$$

Hence,

$$F(\widehat{\Omega}_f) = F(d_{(-,\alpha,U)})\,(f \otimes \mathrm{id}_{U^*})\,b_U = \mathrm{tr}(f).$$

Using this geometric interpretation and following the lines of [Tu2], Section I.2.7 we obtain the next lemma.

5.1.1 Lemma. (i) *For any morphisms* $f\colon U \to V$ *and* $g\colon V \to U$ *in* $\mathcal{C}$*, we have* $\mathrm{tr}(fg) = \mathrm{tr}(gf)$;

(ii) *for any endomorphisms* f *and* g *of objects of* $\mathcal{C}$*, we have* $\mathrm{tr}(f^*) = \mathrm{tr}(f)$ *and* $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$.

5.2 Dimension. We define the *dimension* of an object $U \in \mathcal{C}$ by

$$\dim(U) = \mathrm{tr}(\mathrm{id}_U) = d_{(^U U)}\,c_{U,U^*}\,(\theta_U \otimes \mathrm{id}_{U^*})\,b_U \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

If $U \in \mathcal{C}_\alpha$, then $\dim(U) = F(d_{(-,\alpha,U)})\,b_U$ is the value of F on a colored G-knot represented by a diagram consisting of an embedded oriented circle labeled with (α,U). This knot $K \subset S^3$ is an (oriented) unknot endowed with the homomorphism $\pi_1(C_K) = \mathbb{Z} \to G$ carrying the meridian to α.

The properties of F established in Section 4.7 (or in Lemma 5.1.1) imply that isomorphic objects have equal dimensions and for any $U \in \mathcal{C}$ and $\beta \in G$,

$$\dim(U^*) = \dim(U) \quad \text{and} \quad \dim(\varphi_\beta(U)) = \varphi_\beta(\dim(U)).$$

For any $U, V \in \mathcal{C}$,

$$\dim(U \otimes V) = \dim(U)\dim(V).$$

For morphisms and objects of the neutral component $\mathcal{C}_1 \subset \mathcal{C}$, the definitions above coincide with the standard definitions of trace and dimension in a ribbon category. This implies that for any $f \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$, we have $\mathrm{tr}(f) = f$. In particular, $\dim(\mathbb{1}_{\mathcal{C}}) = \mathrm{tr}(\mathrm{id}_{\mathbb{1}}) = \mathrm{id}_{\mathbb{1}}$.

5.3 Algebra of colors. We define the *algebra of colors* or the *Verlinde algebra* $L = L(\mathcal{C})$ of $\mathcal{C}$. Consider the K-module $\bigoplus_{U \in \mathcal{C}} \mathrm{End}_{\mathcal{C}}(U)$, where U runs over all objects of $\mathcal{C}$. The element of this module represented by $f \in \mathrm{End}_{\mathcal{C}}(U)$ is denoted $\langle U, f\rangle$

or shorter $\langle f\rangle$. We factor this module by the following relations: for any morphisms $f: U \to V$ and $g: V \to U$ in $\mathcal{C}$,

$$\langle V, fg\rangle = \langle U, gf\rangle. \tag{5.3.a}$$

Denote the quotient K-module by L. We provide L with multiplication by the formula $\langle f\rangle\langle f'\rangle = \langle f \otimes f'\rangle$. Clearly, L is an associative K-algebra with unit $\langle \mathrm{id}_{\mathbb{1}}\rangle$. Every object $U \in \mathcal{C}$ determines an element $\langle U\rangle = \langle \mathrm{id}_U\rangle \in L$.

The algebra L is G-graded: $L = \bigoplus_{\alpha\in G} L_\alpha$ where L_α is the submodule of L additively generated by the vectors $\langle U, f\rangle$ with $U \in \mathcal{C}_\alpha$. We have $L_\alpha L_\beta \subset L_{\alpha\beta}$ for all $\alpha, \beta \in G$. The formula $\varphi_\alpha(\langle f\rangle) = \langle\varphi_\alpha(f)\rangle$ defines an action of G on L by algebra automorphisms. Clearly, $\varphi_\alpha(L_\beta) = L_{\alpha\beta\alpha^{-1}}$ for all $\alpha, \beta \in G$. The existence of the braiding implies that $ab = \varphi_\alpha(b)a$ for any $a \in L_\alpha, b \in L$. The existence of the twist implies that $\varphi_\alpha|_{L_\alpha} = \mathrm{id}$ for any $\alpha \in G$. Thus L satisfies the first three axioms (3.1.1)–(3.1.3) (see p. 25) of a crossed G-algebra.

The trace of morphisms defines a K-linear functional $L \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ carrying any generator $\langle f\rangle$ to $\mathrm{tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. We denote this functional by dim. In particular, $\dim(\langle U\rangle) = \mathrm{tr}(\mathrm{id}_U) = \dim(U)$ for any $U \in \mathcal{C}$. It follows from the properties of the trace that dim is an algebra homomorphism.

Sending a generator $\langle U, f\rangle \in L$ to the generator $\langle U^*, f^*: U^* \to U^*\rangle$ we define a K-linear homomorphism $L \to L$ denoted by $*$. It follows from the definitions and Corollary 4.6 that $*$ is an involutive anti-automorphism of L carrying each L_α onto $L_{\alpha^{-1}}$ and commuting with $\dim: L \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$.

5.4 Generalized colorings. The elements of the algebra of colors $L = L(\mathcal{C})$ can be used to color G-links. A *generalized coloring* of a G-link (ℓ, z, g) is a function a which assigns to every path $\gamma: [0,1] \to C_\ell$ from $\gamma(0) = z$ to a point of $\tilde{\ell}$ a vector $a_\gamma \in L_{g(\mu_\gamma)}$ such that

(i) a_γ is preserved under homotopies of γ fixing $\gamma(0)$ and keeping $\gamma(1)$ on $\tilde{\ell}$;

(ii) if β is a loop in (C_ℓ, z), then $a_{\beta\gamma} = \varphi_{g([\beta])}(a_\gamma)$.

A coloring u of (ℓ, z, g) in the sense of Section 3.1 gives rise to a generalized coloring by $a_\gamma = \langle u_\gamma\rangle \in L$.

The invariant F of colored G-links extends to generalized colorings provided we restrict ourselves to a certain class of G-links. A G-link $(\ell = \ell_1 \cup \cdots \cup \ell_n, z, g)$ is *special* if g carries the longitudes of all components of ℓ to $1 \in G$. Let a be a generalized coloring of this G-link. We define $F(\ell, z, g, a)$ as follows. For $i = 1, \ldots, n$, pick a path γ_i from z to a point of $\tilde{\ell}_i$. We transform ℓ into a G-graph Ω by inserting a coupon into each component ℓ_i near $\gamma_i(1)$. Suppose first that for all $i = 1, \ldots, n$, we have $a_{\gamma_i} = \langle U_i, f_i\rangle \in L_{g(\mu_{\gamma_i})}$ with $f_i \in \mathrm{End}(U_i)$. Set $u_{\gamma_i} = U_i$, $v_{\gamma_i} = f_i$ for all i. This extends uniquely to a coloring (u, v) of Ω (it is here that we need ℓ to be special, cf. Lemma 3.2.1). Set $F(\ell, z, g, a) = F(\Omega, g, u, v)$. This does not depend on the choice of the paths γ_i and extends to arbitrary a by additivity. Note that if $a_{\gamma_i} = \langle U_i\rangle$

then there is no need to insert the coupon on ℓ_i; it suffices to keep ℓ_i as a stratum and set $u_{\gamma_i} = U_i$. This gives the same invariant F.

More generally, we call a G-graph *special* if the longitudes of all its circle strata are carried to $1 \in G$. The elements of L can be used to color circle strata of special G-graphs. The invariant F extends to such generalized colorings of G-graphs exactly as above. The properties of F established in Section 4.7 extend to this setting in the obvious way.

Chapter VII
Modular G-categories and HQFTs

VII.1 Modular crossed G-categories

We introduce modular group-categories which will be our main algebraic tools in constructing three-dimensional HQFTs.

1.1 Modular crossed G-categories. Let $\mathcal{C}$ be a crossed G-category over the ring K. An object V of $\mathcal{C}$ is *simple* if $\mathrm{End}_{\mathcal{C}}(V) = K\,\mathrm{id}_V$. It is clear that an object isomorphic or dual to a simple object is itself simple. The action of G on $\mathcal{C}$ transforms simple objects into simple objects.

An object U of $\mathcal{C}$ is *dominated by simple objects* if there exist a finite set of simple objects $\{V_r\}_r$ of $\mathcal{C}$ (possibly with repetitions) and morphisms

$$\{f_r : V_r \to U,\ g_r : U \to V_r\}_r$$

such that $\mathrm{id}_U = \sum_r f_r g_r$. If $U \in \mathcal{C}_\alpha$ with $\alpha \in G$, then without loss of generality we can assume that $V_r \in \mathcal{C}_\alpha$ for all r.

A ribbon crossed G-category $\mathcal{C}$ is *modular* if it satisfies the following five axioms:

(1.1.1) the unit object $\mathbb{1}_{\mathcal{C}}$ is simple;

(1.1.2) for each $\alpha \in G$, the set I_α of the isomorphism classes of simple objects of $\mathcal{C}_\alpha$ is finite;

(1.1.3) for each $\alpha \in G$, any object of $\mathcal{C}_\alpha$ is dominated by simple objects of $\mathcal{C}_\alpha$;

(1.1.4) $\mathrm{Hom}_{\mathcal{C}}(V, W) = 0$ for any non-isomorphic simple objects V, W of $\mathcal{C}$.

To formulate the fifth axiom, we need more notation. For $i, j \in I_1$, choose simple objects $V_i, V_j \in \mathcal{C}_1$ representing i, j, respectively, and set

$$S_{i,j} = \mathrm{tr}(c_{V_j,V_i} \circ c_{V_i,V_j} : V_i \otimes V_j \to V_i \otimes V_j) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = K.$$

It follows from the properties of the trace that $S_{i,j}$ does not depend on the choice of V_i, V_j. Here is the fifth and the last axiom:

(1.1.5) The square matrix $S = [S_{i,j}]_{i,j \in I_1}$ is invertible over K.

These axioms generalize the axioms of a modular ribbon category given in [Tu2] for $G = 1$.

It follows from Axioms (1.1.1)–(1.1.5) that the neutral component $\mathcal{C}_1$ of $\mathcal{C}$ is a modular category in the sense of [Tu2]. By [Tu2], for any objects U, V of $\mathcal{C}_1$, the

K-module $\mathrm{Hom}(U,V)$ is projective of finite type. For objects U, V of $\mathcal{C}$ belonging to the same component of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(U,V)=\mathrm{Hom}_{\mathcal{C}}(\mathbb{1},V\otimes U^*)=\mathrm{Hom}_{\mathcal{C}_1}(\mathbb{1},V\otimes U^*)$$

so that $\mathrm{Hom}_{\mathcal{C}}(U,V)$ is a projective K-module of finite type. If $U,V\in\mathcal{C}$ belong to different components of $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(U,V)=0$. In both cases set

$$\nu_{U,V}=\mathrm{Dim}(\mathrm{Hom}_{\mathcal{C}}(U,V))\in K,$$

where Dim is the dimension of projective K-modules of finite type; see Section I.1.3. We have

$$\nu_{V,U}=\mathrm{Dim}(\mathrm{Hom}_{\mathcal{C}}(V,U))=\mathrm{Dim}(\mathrm{Hom}_{\mathcal{C}}(U,V))=\nu_{U,V},$$

where the second equality follows from Lemma 1.4 below and the fact that the dimensions of dual projective K-modules are equal. Note that for any $U,V,W\in\mathcal{C}$,

$$\nu_{U,V\otimes W}=\nu_{U\otimes W^*,V}=\nu_{V,U\otimes W^*}. \tag{1.1.a}$$

Axiom (1.1.1) and the identity $\varphi_\alpha(\mathrm{id}_{\mathbb{1}})=\mathrm{id}_{\mathbb{1}}$ imply that all $\alpha\in G$ act on $\mathrm{End}_{\mathcal{C}}(\mathbb{1})=K$ as the identity. Therefore the dimension of objects of $\mathcal{C}$ is invariant under the action of G: for any $V\in\mathcal{C}$ and $\alpha\in G$,

$$\dim(\varphi_\alpha(V))=\varphi_\alpha(\dim(V))=\dim(V).$$

If the ground ring K is a field, then Axiom (1.1.4) is redundant. It is easy to show (using, for instance, Lemma 1.4 below) that in this case any non-zero morphism between simple objects is an isomorphism.

We need three lemmas concerning a modular crossed G-category $\mathcal{C}$ over K. The first lemma computes the algebra of colors $L(\mathcal{C})=\bigoplus_{\alpha\in G}L_\alpha$ as a K-module.

1.2 Lemma. *Let $\alpha\in G$ and $\{V_i^\alpha\in\mathcal{C}_\alpha\}_{i\in I_\alpha}$ be representatives of the isomorphism classes of simple objects of $\mathcal{C}_\alpha$. Then L_α is a free K-module with basis $\{\langle V_i^\alpha\rangle\}_{i\in I_\alpha}$. For any $U\in\mathcal{C}_\alpha$,*

$$\langle U\rangle=\sum_{i\in I_\alpha}\nu_{V_i^\alpha,U}\,\langle V_i^\alpha\rangle, \tag{1.2.a}$$

$$\dim(U)=\sum_{i\in I_\alpha}\nu_{V_i^\alpha,U}\,\dim(V_i^\alpha). \tag{1.2.b}$$

Proof. Let $\langle U\in\mathcal{C}_\alpha, f:U\to U\rangle$ be a generator of L_α. By (1.1.3), there is a finite system of objects $\{V_{i(r)}\}_r$ belonging to the family $\{V_i^\alpha\}_{i\in I_\alpha}$ (possibly with repetitions) and morphisms $\{f_r:V_{i(r)}\to U, g_r:U\to V_{i(r)}\}_r$ such that $\mathrm{id}_U=\sum_r f_rg_r$. Then $f=\sum_r ff_rg_r$ and

$$\langle U,f\rangle=\sum_r\langle U,ff_rg_r\rangle=\sum_r\langle V_{i(r)},g_rff_r\rangle.$$

Since $V_{i(r)}$ is a simple object, $g_r f f_r = k_r \,\mathrm{id}_{V_{i(r)}}$ for some $k_r \in K$. Hence

$$\langle U, f\rangle = \sum_r \langle V_{i(r)}, k_r \mathrm{id}_{V_{i(r)}}\rangle = \sum_r k_r \langle V_{i(r)}, \mathrm{id}_{V_{i(r)}}\rangle = \sum_r k_r \langle V_{i(r)}\rangle.$$

Therefore the vectors $\{\langle V_i^\alpha\rangle\}_{i\in I_\alpha}$ generate L_α over K. To prove their linear independence, we define for each $i \in I_\alpha$ a linear functional $t_i : L(\mathcal{C}) \to K$ as follows. Let $\langle U \in \mathcal{C}, f : U \to U\rangle$ be a generator of L. Denote by f_i the automorphism of the K-module $\mathrm{Hom}_{\mathcal{C}}(V_i^\alpha, U)$ carrying each $h \in \mathrm{Hom}_{\mathcal{C}}(V_i^\alpha, U)$ into fh. Set $t_i(f) = \mathrm{Tr}(f_i) \in K$. It follows from the standard properties of the trace that t_i annihilates the relation given in (5.3.a), Chapter VI, and defines thus a linear functional $t_i : L(\mathcal{C}) \to K$. By Axiom (1.1.4), $t_i(\langle V_j^\alpha\rangle) = \delta_j^i \in K$, where δ_j^i is the Kronecker delta. This implies that L_α is a free K-module with basis $\{\langle V_i^\alpha\rangle\}_{i\in I_\alpha}$.

For any object $U \in \mathcal{C}_\alpha$ and any $f \in \mathrm{End}_{\mathcal{C}}(U)$, there is a unique decomposition $\langle U, f\rangle = \sum_{i\in I_\alpha} r_i \langle V_i^\alpha\rangle$ with $r_i \in K$. Applying t_i, we obtain $r_i = t_i(f) = \mathrm{Tr}(f_i)$ for all i. For $f = \mathrm{id}_U$, this gives

$$r_i = t_i(\mathrm{id}_U) = \mathrm{Dim}(\mathrm{Hom}_{\mathcal{C}}(V_i^\alpha, U)) = \nu_{V_i^\alpha, U}.$$

This implies (1.2.a). Applying the functional $\dim : L(\mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = K$ to both sides of (1.2.a), we obtain (1.2.b). □

1.3 Corollary. $L_\alpha = 0$ *if and only if the category* $\mathcal{C}_\alpha$ *is void.*

By a *void category* $\emptyset$, we mean a category with empty classes of objects and morphisms. Clearly, if $\mathcal{C}_\alpha = \emptyset$, then $L_\alpha = 0$. Conversely, if $L_\alpha = 0$ then by Lemma 1.2, the category $\mathcal{C}_\alpha$ has no simple objects. By (1.1.3), it is void.

1.4 Lemma. *For any objects* U, V *of* $\mathcal{C}$, *the pairing*

$$\mathrm{Hom}_{\mathcal{C}}(U, V) \otimes_K \mathrm{Hom}_{\mathcal{C}}(V, U) \to K, \ (x, y) \mapsto \mathrm{tr}(yx),$$

is non-degenerate.

Proof. Assume first that $U = \mathbb{1} \in \mathcal{C}_1$. If $V \in \mathcal{C}_1$, then the claim of the lemma follows from the standard properties of the modular category $\mathcal{C}_1$. If $V \in \mathcal{C}_\alpha$ with $\alpha \neq 1$, then the claim of the lemma is obvious since $\mathrm{Hom}_{\mathcal{C}}(U, V) = \mathrm{Hom}_{\mathcal{C}}(V, U) = 0$. For an arbitrary $U \in \mathcal{C}$, we have canonical isomorphisms

$$\alpha : \mathrm{Hom}_{\mathcal{C}}(U, V) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V \otimes U^*), \quad \beta : \mathrm{Hom}_{\mathcal{C}}(V, U) \to \mathrm{Hom}_{\mathcal{C}}(V \otimes U^*, \mathbb{1}),$$

such that $\mathrm{tr}(yx) = \mathrm{tr}(\beta(y)\alpha(x))$ for any $x \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ and $y \in \mathrm{Hom}_{\mathcal{C}}(V, U)$, cf. the proof of Lemma II.4.2.3 in [Tu2]. Therefore the general case of the lemma follows from the case $U = \mathbb{1}$. □

1.5 Lemma. *For any simple object* V *of* $\mathcal{C}$, *the dimension* $\dim(V) \in K$ *is invertible in* K.

Proof. With respect to the generator $\mathrm{id}_V \in \mathrm{Hom}_{\mathcal{C}}(V, V) = K$ the bilinear form $(x, y) \mapsto \mathrm{tr}(yx) \in K$ on $\mathrm{Hom}_{\mathcal{C}}(V, V)$ is presented by the (1×1)-matrix $[\mathrm{tr}(\mathrm{id}_V)] = [\dim(V)]$. The non-degeneracy of this form implies that $\dim(V) \in K^*$. □

1.6 The canonical color. Let $\mathcal{C}$ be a modular crossed G-category over K. For $\alpha \in G$, choose representatives $\{V_i^\alpha \in \mathcal{C}_\alpha\}_{i \in I_\alpha}$ of the isomorphism classes of simple objects of the category $\mathcal{C}_\alpha$. We define a *canonical color* $\omega_\alpha \in L_\alpha$ by

$$\omega_\alpha = \sum_{i \in I_\alpha} \dim(V_i^\alpha) \langle V_i^\alpha \rangle. \tag{1.6.a}$$

It is clear that ω_α does not depend on the choice of representatives $\{V_i^\alpha\}_{i \in I_\alpha}$. Lemmas 1.2 and 1.5 imply that $\omega_\alpha = 0$ if and only if the category $\mathcal{C}_\alpha$ is void.

The fact that the duality $V \mapsto V^*$ transforms simple objects of $\mathcal{C}_\alpha$ into simple objects of $\mathcal{C}_{\alpha^{-1}}$ and preserves the dimension implies that for all $\alpha \in G$,

$$(\omega_\alpha)^* = \omega_{\alpha^{-1}}, \tag{1.6.b}$$

where $*$ is the involution on L defined in Section VI.5.3. Since the action of $\alpha \in G$ on $\mathcal{C}$ transforms simple objects in any $\mathcal{C}_\beta$ into simple objects in $\mathcal{C}_{\alpha\beta\alpha^{-1}}$, it induces a dimension-preserving bijection $I_\beta \to I_{\alpha\beta\alpha^{-1}}$. Therefore

$$\varphi_\alpha(\omega_\beta) = \omega_{\alpha\beta\alpha^{-1}}. \tag{1.6.c}$$

1.6.1 Lemma. *For any $\alpha, \beta \in G$ and $V \in \mathcal{C}_\beta$,*

$$\omega_\alpha \langle V \rangle = \dim(V)\, \omega_{\alpha\beta} \quad \textit{and} \quad \langle V \rangle\, \omega_\alpha = \dim(V)\, \omega_{\beta\alpha}.$$

Proof. We prove only the first equality, the second one is similar. We follow the argument given in [Bru] for $G = 1$. We have

$$\begin{aligned}
\omega_\alpha \langle V \rangle &= \sum_{i \in I_\alpha} \dim(V_i^\alpha) \langle V_i^\alpha \rangle \langle V \rangle \\
&= \sum_{i \in I_\alpha} \dim(V_i^\alpha) \langle V_i^\alpha \otimes V \rangle \\
&= \sum_{i \in I_\alpha} \sum_{j \in I_{\alpha\beta}} \dim(V_i^\alpha)\, \nu_{V_j^{\alpha\beta}, V_i^\alpha \otimes V} \langle V_j^{\alpha\beta} \rangle \\
&= \sum_{i \in I_\alpha} \sum_{j \in I_{\alpha\beta}} \dim(V_i^\alpha)\, \nu_{V_i^\alpha, V_j^{\alpha\beta} \otimes V^*} \langle V_j^{\alpha\beta} \rangle \\
&= \sum_{j \in I_{\alpha\beta}} \Big(\sum_{i \in I_\alpha} \nu_{V_i^\alpha, V_j^{\alpha\beta} \otimes V^*} \dim(V_i^\alpha) \Big) \langle V_j^{\alpha\beta} \rangle \\
&= \sum_{j \in I_{\alpha\beta}} \dim(V_j^{\alpha\beta} \otimes V^*) \langle V_j^{\alpha\beta} \rangle \\
&= \sum_{j \in I_{\alpha\beta}} \dim(V_j^{\alpha\beta})\, \dim(V^*) \langle V_j^{\alpha\beta} \rangle \\
&= \dim(V^*)\, \omega_{\alpha\beta} \\
&= \dim(V)\, \omega_{\alpha\beta},
\end{aligned}$$

where among others we use equalities (1.1.a), (1.2.a), and (1.2.b). □

1.7 Elements $\mathcal{D}$, $\Delta_\pm$ of K. We shall need several elements of K associated with the neutral component $\mathcal{C}_1$ of a modular crossed G-category $\mathcal{C}$. Let $\{V_i\}_{i \in I_1}$ be representatives of the isomorphism classes of simple objects in $\mathcal{C}_1$. A *rank* of $\mathcal{C}_1$ is an element $\mathcal{D} \in K$ such that

$$\mathcal{D}^2 = \sum_{i \in I_1} (\dim(V_i))^2 \in K. \tag{1.7.a}$$

The existence of a rank is a minor technical condition which does not reduce the range of our constructions.

Since each $V_i \in \mathcal{C}_1$ is a simple object, the twist $\theta_{V_i} : V_i \to \varphi_1(V_i) = V_i$ equals $v_i \, \mathrm{id}_{V_i}$ for some $v_i \in K$. Since θ_{V_i} is invertible, $v_i \in K^*$. Set

$$\Delta_\pm = \sum_{i \in I_1} v_i^{\pm 1} (\dim(V_i))^2 \in K.$$

We can interpret $\Delta_\pm \in K$ as the invariant F of an unknot $\ell \subset S^3$ endowed with framing ± 1, trivial homomorphism $\pi_1(C_\ell) \to \{1\} \subset G$ and (generalized) color $\omega_1 \in L_1$. It is known that $\mathcal{D}$ and $\Delta_\pm$ are invertible in K and that $\Delta_+ \Delta_- = \mathcal{D}^2$; see [Tu2], formula (II.2.4.a).

For any $\alpha \in G$, set

$$d_\alpha = \dim(\omega_\alpha) = \sum_{i \in I_\alpha} (\dim(V_i^\alpha))^2 \in K, \tag{1.7.b}$$

where I_α is the set of isomorphism classes of simple objects in $\mathcal{C}_\alpha$ and V_i^α is a simple object of $\mathcal{C}_\alpha$ representing $i \in I_\alpha$. In particular, $d_1 = \mathcal{D}^2$. We can compute d_α for all $\alpha \in G$ as follows. If the category $\mathcal{C}_\alpha$ is void, then $I_\alpha = \emptyset$ and $d_\alpha = 0$. If $\mathcal{C}_\alpha \neq \emptyset$, then $d_\alpha = d_1 = \mathcal{D}^2$. Indeed, if $\mathcal{C}_\alpha \neq \emptyset$, then $\mathcal{C}_{\alpha^{-1}} \neq \emptyset$ and there is at least one simple object V in $\mathcal{C}_{\alpha^{-1}}$. By Lemma 1.6.1, $\langle V \rangle \, \omega_\alpha = \dim(V) \, \omega_1$ and therefore

$$\dim(V) \, d_\alpha = \dim(\langle V \rangle \, \omega_\alpha) = \dim(\dim(V) \, \omega_1) = \dim(V) \, d_1.$$

Now Lemma 1.5 implies that $d_\alpha = d_1$.

1.8 Examples and constructions of modular crossed G-categories. The ribbon crossed G-category $\mathcal{C} = \mathcal{C}(a, b, c, \theta)$ discussed in Section VI.2.6 is modular by the obvious reasons: each category $\mathcal{C}_\alpha$ has only one object, this object is simple, and the matrix $S = [1]$ is the unit (1×1)-matrix. If $H \subset G$ is a finite subgroup of the center of G, then pushing $\mathcal{C}(a, b, c, \theta)$ forward along the projection $G \to G/H$ we obtain a ribbon crossed (G/H)-category satisfying Axioms (1.1.1)–(1.1.4) but possibly not (1.1.5). This is a special case of the following fact: if the kernel of a group epimorphism $q : G' \to G$ is finite and acts as the identity on a modular crossed G'-category $\mathcal{C}'$, then the push-forward G-category $q_*(\mathcal{C}')$ (cf. Sections VI.1.4, VI.2.5) satisfies all axioms of a modular crossed G-category except possibly (1.1.5). On the other hand, the pull-back of any modular crossed G-category $\mathcal{C}$ along a group homomorphism $G' \to G$ is always a modular crossed G'-category.

A tensor product of a finite family of modular crossed G-categories is always modular. A direct product of $n \geq 2$ modular crossed G-categories is not modular because the unit object in the product is not simple, $\mathrm{End}(\mathbb{1}) = K^n$. The mirror image of a modular crossed G-category is modular, cf. [Tu2], Exercise II.1.9.2.

Examples of modular crossed G-categories with $G = \mathbb{Z}/2\mathbb{Z}$ are provided by the categories of representations of $U_q(sl_2(\mathbb{C}))$ at roots of unity. We use a topological description of these categories given in [Tu2] (cf. [Th]). Let $r \geq 3$ be an odd integer and a be a primitive complex root of unity of order $4r$. In [Tu2] the author used tangles and Jones–Wenzl idempotents to define a ribbon category $\mathcal{V}(a)$ whose objects are finite sequences $(j_1, \dots, j_l)$ of integers belonging to the set $\{1, 2, \dots, r-2\}$. We label such an object with $j_1 + \dots + j_l \pmod 2 \in \mathbb{Z}/2\mathbb{Z} = G$. The category $\mathcal{V}(a)$ is a disjoint union of two full subcategories comprising objects labeled by 0 and 1, respectively. The crossing isomorphisms $\{\varphi_\alpha\}_{\alpha \in G}$ are the identity maps. The standard ribbon structure on $\mathcal{V}(a)$ turns $\mathcal{V}(a)$ into a ribbon crossed G-category. Its simple objects are the same as in the standard theory. The results of [Tu2], Section XII.7.5 show that if r is odd, then $\mathcal{V}(a)$ a modular crossed G-category.

1.9 Remark. One can introduce the concepts of Hermitian and unitary modular crossed G-categories extending the known definitions for $G = 1$. Such categories give rise to Hermitian and unitary 3-dimensional HQFTs, respectively.

1.10 Exercise. Let $\mathcal{C}$ be a modular crossed G-category over K. Show that the set $\{\alpha \in G \mid \mathcal{C}_\alpha \neq \emptyset\}$ is a normal subgroup of G. Hint: use Lemma 1.6.1.

VII.2 Invariants of 3-dimensional G-manifolds

Throughout the rest of this chapter, we fix a modular crossed G-category $\mathcal{C}$ over the ring K and an Eilenberg–MacLane CW-space $X = K(G, 1)$ with base point x. In the sequel, $L = \bigoplus_{\alpha \in G} L_\alpha$ is the algebra of colors of $\mathcal{C}$, $\mathcal{D} \in K^*$ is a rank of $\mathcal{C}$, and Δ_+, Δ_- are the elements of K^* defined in Section 1.7.

In this section we derive from $\mathcal{C}$ a topological invariant of three-dimensional X-manifolds. To stress the independence of the resulting theory of the choice of X, we shall call these manifolds G-manifolds.

2.1 G-manifolds. By a *G-manifold*, we mean a pair (W, g), where W is a (smooth) manifold and g is a free homotopy class of maps from W to $X = K(G, 1)$. Recall that such g bijectively correspond to isomorphism classes of principal G-bundles over W. A G-manifold (W, g) is m-dimensional (resp. closed, connected, oriented, etc.) if W is m-dimensional (resp. closed, connected, oriented, etc.). A *homeomorphism of G-manifolds* $(W, g) \to (W', g')$ is a diffeomorphism $f : W \to W'$ such that $g = g'f$. We will work with oriented manifolds and consider only orientation preserving homeomorphisms.

2.2 Invariant $\tau_{\mathcal{C}}$ of 3-dimensional G-manifolds. Let (W, g) be a closed connected oriented 3-dimensional G-manifold. We derive from the category $\mathcal{C}$ a homeomorphism invariant $\tau_{\mathcal{C}}(W, g) \in K$ as follows. Present W as the result of surgery on S^3 along a framed link ℓ with $\#\ell$ components. Recall that W is obtained by gluing $\#\ell$ solid tori to the exterior $E = E_\ell$ of ℓ in S^3. Take any point $z \in E \subset W$. Pick in the homotopy class g a map $W \to X$ carrying z to the base point x of X. The restriction of this map to E induces a homomorphism

$$\pi_1(E_\ell, z) = \pi_1(E, z) \to \pi_1(X, x) = G$$

denoted by the same letter g. We fix an arbitrary orientation of ℓ. The triple (ℓ, z, g) is a special G-link in the sense of Section VI.5.4. We provide (ℓ, z, g) with the following (generalized) coloring. To every path γ in C_ℓ from z to $\tilde{\ell}$, we assign the canonical color $\omega_{g(\mu_\gamma)} \in L_{g(\mu_\gamma)}$. By (1.6.c), this satisfies conditions (i) and (ii) of Section VI.5.4 and defines a *canonical coloring* of (ℓ, z, g). Denote the resulting colored G-link by ℓ_{can}. Set

$$\tau_{\mathcal{C}}(W, g) = \Delta_-^{\sigma(\ell)} \mathcal{D}^{-\sigma(\ell)-\#\ell-1} F(\ell_{\mathrm{can}}) \in K, \tag{2.2.a}$$

where $\sigma(\ell)$ is the signature of the compact oriented 4-manifold B_ℓ obtained from the closed 4-ball B^4 by attaching 2-handles along tubular neighborhoods of the components of the framed link $\ell \subset S^3 = \partial B^4$. Here $\partial B_\ell = W$ and the orientation of B_ℓ is induced by the one of W.

2.3 Theorem. *$\tau_{\mathcal{C}}(W, g)$ is a homeomorphism invariant of (W, g).*

Proof. We should prove that $\tau_{\mathcal{C}}(W, g)$ does not depend on the choices made in its definition. First of all, the homomorphism g is defined only up to conjugation. It follows from (1.6.c) that the colored G-links ℓ_{can} corresponding to g and $\alpha g \alpha^{-1}$ with $\alpha \in G$ are related by the transformation φ_α defined in the proof of Lemma VI.3.4.1. By Theorem VI.3.6 and Condition VI.3.5(v), the corresponding values $F(\ell_{\mathrm{can}}) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = K$ are also related via φ_α. Since φ_α acts on $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ as the identity, $F(\ell_{\mathrm{can}})$ does not depend on the choice of g in its conjugacy class.

The independence of the choice of z follows from the invariance of $F(\ell_{\mathrm{can}})$ under transfers along paths discussed in Section VI.3.1.

Let ℓ' be obtained from ℓ by reversing the orientation on one of the components. By (1.6.b), ℓ_{can} and ℓ'_{can} are related by the transformation described in Section VI.4.7.2. Therefore $F(\ell_{\mathrm{can}}) = F(\ell'_{\mathrm{can}})$ and $\tau_{\mathcal{C}}(W, g)$ does not depend on the choice of orientation on ℓ.

To prove the independence of the choice of ℓ we use the Kirby moves on links. According to Kirby [Ki], any two framed links in S^3 yielding through surgery homeomorphic 3-manifolds can be related by isotopy and certain transformations, called the Kirby moves. There are moves of two kinds. The first move adds to a framed link $\ell \subset S^3$ a separated unknot $\ell^\pm$ with framing ± 1; under this move the 4-manifold B_ℓ is transformed into $B_\ell \# \mathbb{C}P^2$. The second move preserves B_ℓ and is induced by a sliding of a 2-handle of B_ℓ across another 2-handle. We need a more precise version of this theory. For a framed link $\ell \subset S^3$, let $W_\ell = \partial B_\ell$ be the result of surgery on ℓ. A *surgery presentation* of a closed connected oriented 3-manifold W is a pair (a framed link $\ell \subset S^3$, an isotopy class of orientation preserving homeomorphisms $f: W \to W_\ell$). Note that any isotopy of ℓ into a framed link $\ell' \subset S^3$ induces a homeomorphism $j_{\ell,\ell'}: W_\ell \to W_{\ell'}$. The pair $(\ell', j_{\ell,\ell'} f)$ also is a surgery presentation of W; we say that it is obtained from (ℓ, f) by isotopy. The first Kirby move $\ell \mapsto \ell' = \ell \amalg \ell^\pm$ induces a homeomorphism $j_{\ell,\ell'}: W_\ell \to W_{\ell'}$ which is the identity outside a small 3-ball in $S^3 - \ell$ containing $\ell^\pm$. The second Kirby move $\ell \mapsto \ell'$ induces a homeomorphism $B_\ell \to B_{\ell'}$ which restricts to a homeomorphism of boundaries $j_{\ell,\ell'}: W_\ell \to W_{\ell'}$. In both cases, we say that the surgery presentation $(\ell', j_{\ell,\ell'} f: W \to W_{\ell'})$ is obtained from $(\ell, f: W \to W_\ell)$ by the Kirby move. The arguments in [Ki], Section 2 show that for any surgery presentations $(\ell_1, f_1: W \to W_{\ell_1})$ and $(\ell_2, f_2: W \to W_{\ell_2})$ of a closed connected oriented 3-manifold W, there is a sequence of isotopies and Kirby moves transforming (ℓ_1, f_1) into (ℓ_2, f_2).

The 3-manifold W_ℓ obtained by surgery on a special G-link $(\ell \subset S^3, g: \pi_1(C_\ell) \to G)$ is a G-manifold in the obvious way. An isotopy or a Kirby move on a special G-link (ℓ, g) yields a special G-link $(\ell' \subset S^3, g': \pi_1(C_{\ell'}) \to G)$, where g' is the composition of the inclusion homomorphism $\pi_1(C_{\ell'}) \to \pi_1(W_{\ell'})$, the isomorphism $\pi_1(W_{\ell'}) \approx \pi_1(W_\ell)$ induced by $j_{\ell,\ell'}: W_\ell \approx W_{\ell'}$, and the homomorphism $\pi_1(W_\ell) \to G$ induced by g. The results of the previous paragraph imply that if surgeries on two special G-links in S^3 yield homeomorphic G-manifolds, then these G-links can be related by a finite sequence of isotopies and Kirby moves.

It is clear that $\tau_{\mathcal{C}}(W_\ell, g)$ is invariant under isotopy of ℓ. To prove the theorem it is thus enough to show that $\tau_{\mathcal{C}}(W_\ell, g)$ is invariant under the Kirby moves on ℓ. Consider the first Kirby move $\ell \mapsto \ell' = \ell \amalg \ell^\pm$. The meridian of $\ell^\pm$ is contractible in $W_{\ell'}$ and therefore the colored G-link ℓ'_{can} is a disjoint union of ℓ_{can} and the (± 1)-framed unknot $\ell^\pm_{\mathrm{can}}$ endowed with the trivial homomorphism $\pi_1(S^3 - \ell^\pm_{\mathrm{can}}) \to \{1\} \subset G$ and the color $\omega_1 \in L_1$. We have

$$F(\ell'_{\mathrm{can}}) = F(\ell^\pm_{\mathrm{can}})\, F(\ell_{\mathrm{can}}) = \Delta_\pm\, F(\ell_{\mathrm{can}}).$$

This implies the invariance of $\tau_{\mathcal{C}}(W, g)$ under the first Kirby move, since

$$\#\ell' = \#\ell + 1, \quad \sigma(\ell') = \sigma(\ell) \pm 1, \quad \Delta_+\Delta_- = \mathcal{D}^2.$$

We consider the second Kirby moves in the restricted form studied by Fenn and Rourke [FR1]. The Kirby–Fenn–Rourke moves split into positive and negative ones. It is explained in [RT] that (modulo the first Kirby moves) it is enough to consider only the negative Kirby–Fenn–Rourke moves. Such a move $\ell \mapsto \ell'$ replaces a piece T of ℓ lying in a closed 3-ball $B^3 \subset S^3$ by another piece T' lying in B^3 and having the same endpoints. Here $T = B^3 \cap \ell$ is a system of $k \geq 1$ parallel strings with parallel framings and $T' = T^- \amalg t$, where T^- is obtained from T by applying a full left-hand twist and t is an unknot encircling T and having the framing -1. Note that

$\#\ell' = \#\ell + 1$ and $\sigma(\ell') = \sigma(\ell) - 1$. We must prove that

$$F(\ell'_{\mathrm{can}}) = \Delta_- \, F(\ell_{\mathrm{can}}). \tag{2.3.a}$$

This equality follows from a "local" equality involving only T and T'. To formulate this local equality, we position T as a trivial braid in $\mathbb{R}^2 \times [0,1]$ with constant framing. The framed tangle $T' = T^- \cup t \subset \mathbb{R}^2 \times [0,1]$ is obtained from T as above. Note that $C_T = (\mathbb{R}^2 \times [0,1]) - T$ is obtained from $C_{T^-} = (\mathbb{R}^2 \times [0,1]) - T^-$ by surgery on $t \subset C_{T^-}$. Therefore any homomorphism $g\colon \pi_1(C_T) \to G$ induces a homomorphism $g'\colon \pi_1(C_{T'}) \to G$ carrying the homotopy class of the (-1)-longitude of t to $1 \in G$. Given an orientation of T and t (and the induced orientation on T^-), any coloring u of (T, g) induces a coloring u' of (T', g') such that the sources and targets of T and T' coincide and the circle $t \subset T'$ has the canonical color. To prove (2.3.a), it is enough to show that for any orientation of $T \cup t$, any homomorphism $g\colon \pi_1(C_T) \to G$, and any coloring u of (T, g),

$$F(T', g', u') = \Delta_- \, F(T, g, u). \tag{2.3.b}$$

Let us prove this formula. Let $(\varepsilon_1, \alpha_1, U_1), \dots, (\varepsilon_k, \alpha_k, U_k)$ be the source of T. Using the standard technique of coupons colored with identity morphisms, we can reduce the general case to the case where T is a single string oriented from top to bottom and colored with the object $\otimes_{r=1}^{k} (U_r)^{\varepsilon_r}$ of $\mathcal{C}$. Using a decomposition of the identity endomorphism of this object provided by (1.1.3), we can further reduce ourselves to the case where the string T is colored with a simple object V of $\mathcal{C}$. Then the source and target of T (and T') are a 1-term sequence $(+, \alpha, V)$, where $\alpha \in G$ and $V \in \mathcal{C}_\alpha$. By the argument above in this proof, $F(T', g', u')$ does not change if we invert the orientation of t. Therefore we can assume that t is oriented so that its linking number with the string T is equal to -1. Clearly, $F(T) = \mathrm{id}_V$. Since V is simple, $F(T') = a \, \mathrm{id}_V$ for some $a \in K$. To establish (2.3.b), we need to prove that $a = \Delta_-$. Consider the closure $\widehat{T}'$ of T'. This a colored 2-component G-link. As in the standard theory (cf. [Tu2], Corollary I.2.7.1),

$$F(\widehat{T}') = \mathrm{tr}(F(T')) = \mathrm{tr}(a \, \mathrm{id}_V) = a \, \mathrm{tr}(\mathrm{id}_V) = a \, \dim(V). \tag{2.3.c}$$

On the other hand, it is clear that the colored G-link $\widehat{T}'$ can be obtained by doubling (see Section VI.4.7.3) from an unknot $\ell^- \subset S^3$ endowed with framing -1, trivial homomorphism $\pi_1(C_{\ell^-}) \to \{1\} \subset G$, and color $\omega_{\alpha^{-1}} V \in L_1$. By Lemma 1.6.1 and Section 1.7, the invariant F of this colored G-unknot is computed by

$$F(\ell^-, \omega_{\alpha^{-1}} V) = F(\ell^-, \dim(V)\, \omega_1) = \dim(V)\, F(\ell^-, \omega_1) = \dim(V)\, \Delta_-.$$

Since F is preserved under doubling,

$$F(\widehat{T}') = F(\ell^-, \omega_{\alpha^{-1}} V) = \dim(V)\, \Delta_- = \Delta_- \, \dim(V).$$

Comparing with (2.3.c) and using Lemma 1.5, we obtain that $a = \Delta_-$. This proves (2.3.b), (2.3.a), and the theorem. □

2.4 Computations and remarks. 1. The 3-sphere S^3 is simply connected and admits a unique structure of a G-manifold. Presenting S^3 as the result of surgery on S^3 along an empty link we obtain $\tau_{\mathcal{C}}(S^3) = \mathcal{D}^{-1}$.

2. For each $\alpha \in G$, there is a unique free homotopy class of maps $g_\alpha\colon S^1 \times S^2 \to X = K(G,1)$ whose restriction to $S^1 \times \mathrm{pt}$ represents α. Then

$$\tau_{\mathcal{C}}(S^1 \times S^2, g_\alpha) = \begin{cases} 0 & \text{if the category } \mathcal{C}_\alpha \text{ is void,} \\ 1 & \text{otherwise.} \end{cases} \tag{2.4.a}$$

Indeed, we can obtain $(S^1 \times S^2, g_\alpha)$ by surgery on S^3 along a G-unknot ℓ with framing 0 and with homomorphism $\pi_1(C_\ell) \to G$ carrying a meridian of ℓ into $\alpha \in G$. By definition, $\sigma(\ell) = 0$ and

$$\tau_{\mathcal{C}}(S^1 \times S^2, g_\alpha) = \mathcal{D}^{-2} F(\ell_{\mathrm{can}}) = \mathcal{D}^{-2} d_\alpha,$$

where $d_\alpha \in K$ is defined by (1.7.b). The computation of d_α in Section 1.7 implies (2.4.a).

3. Formula (2.2.a) can be rewritten in a more symmetric form:

$$\tau_{\mathcal{C}}(W, g) = \mathcal{D}^{-b_1(W)-1} \Delta_-^{-\sigma_-} \Delta_+^{-\sigma_+} F(\ell_{\mathrm{can}}),$$

where $b_1(W) = \#\ell - \sigma_+ - \sigma_-$ is the first Betti number of W and σ_+ (resp. σ_-) is the number of positive (resp. negative) squares in the diagonal decomposition of the intersection form $H_2(B_\ell; \mathbb{R}) \times H_2(B_\ell; \mathbb{R}) \to \mathbb{Z}$. The invariant

$$\tau'_{\mathcal{C}}(W, g) = \Delta_-^{-\sigma_-} \Delta_+^{-\sigma_+} F(\ell_{\mathrm{can}}) = \mathcal{D}^{b_1(W)+1} \tau_{\mathcal{C}}(W, g)$$

does not depend on the choice of $\mathcal{D}$. Note that $\tau'_{\mathcal{C}}(W, g)$ is defined for a wider class of ribbon crossed G-categories $\mathcal{C}$ satisfying (1.1.1)–(1.1.4) and such that $\Delta_+, \Delta_- \in K^*$. The latter condition is a weakened form of (1.1.5): it follows from (1.1.1)–(1.1.5) but in general does not imply (1.1.5). Condition (1.1.5) is needed in the construction of a 3-dimensional HQFT below.

4. It is easy to deduce from the definitions that for any closed connected oriented 3-dimensional G-manifolds W_1, W_2,

$$\tau_{\mathcal{C}}(W_1 \# W_2) = \mathcal{D}\, \tau_{\mathcal{C}}(W_1)\, \tau_{\mathcal{C}}(W_2),$$

where the structure of a G-manifold on $W_1 \# W_2$ is induced by those on W_1, W_2. Therefore the invariant $\mathcal{D}\, \tau_{\mathcal{C}}$ is multiplicative with respect to the connected sum.

5. We extend $\tau_{\mathcal{C}}(W, g) \in K$ to non-connected closed oriented 3-dimensional G-manifolds by multiplicativity so that

$$\tau_{\mathcal{C}}(W \amalg W', g) = \tau_{\mathcal{C}}(W, g|_W)\, \tau_{\mathcal{C}}(W', g|_{W'}).$$

6. Free homotopy classes of maps $W \to K(\mathbb{Z}/2\mathbb{Z}, 1)$ bijectively correspond to cohomology classes $\xi \in H^1(W; \mathbb{Z}/2\mathbb{Z})$. Let $\mathcal{C} = \mathcal{V}(a)$ be the modular $(\mathbb{Z}/2\mathbb{Z})$-category of Section 1.8 associated with a primitive complex root of unity a of order $4r$ with odd $r \geq 3$. The corresponding invariant $\tau_{\mathcal{C}}(W, \xi)$ was first introduced in [Bl], [KM], [Tu1]. For $\xi = 0$, this is the quantum SO(3)-invariant of W.

2.5 Extended G-manifolds. The invariant $\tau_{\mathcal{C}}$ defined above can be generalized to 3-manifolds with partial G-structure, i.e., with a principal G-bundle on the complement of a framed oriented link or more generally on the complement of a ribbon graph. This graph should be $\mathcal{C}$-colored. We consider here only closed 3-manifolds, for the case of 3-manifolds with boundary, see Section 5. We proceed to precise definitions.

Let W be a closed connected oriented 3-manifold. A *ribbon graph* in W consists of a finite number of framed oriented embedded segments, circles and coupons which are disjoint, except that the endpoints of the segments lie on the bases of the coupons, and the framing satisfies the same conditions as in Section VI.4.1. Since W is connected, the complement of a ribbon graph in W is connected. A G-*graph* in W is a ribbon graph $\Omega \subset W$ whose complement $C_\Omega = W - \Omega$ is endowed with a base point z and a homomorphism $\pi_1(C_\Omega, z) \to G$. This homomorphism may extend or not to $\pi_1(W)$. In the sequel, we shall make no distinction between homomorphisms $\pi_1(C_\Omega, z) \to G$ and maps $C_\Omega \to X$ carrying z to the base point x of X and considered up to homotopy. A G-graph without coupons is a G-link as defined in Section VI.3.1.

A G-graph in W is *colored* (over $\mathcal{C}$) if it is equipped with two functions u, v as in Section VI.4.1. Ambient isotopies of W in itself apply to G-graphs and colored G-graphs in W in the obvious way. This allows us to consider the (ambient) isotopy classes of such graphs. As in Section VI.3.1, we can transfer the structure of a colored G-graph Ω along paths in C_Ω relating various base points. The transfer preserves the ambient isotopy class of a colored G-graph.

A pair consisting of a closed connected oriented 3-manifold W and a colored G-graph in W is called a *connected* 3-*dimensional extended G-manifold without boundary*. Let (W, Ω), (W', Ω') be two such pairs. Here

$$\Omega = (\Omega \subset W, z \in C_\Omega = W - \Omega, g\colon \pi_1(C_\Omega, z) \to G, u, v) \tag{2.5.a}$$

is a colored G-graph in W and

$$\Omega' = (\Omega' \subset W', z' \in C_{\Omega'}, g'\colon \pi_1(C_{\Omega'}, z') \to G, u', v')$$

is a colored G-graph in W'. An *e-homeomorphism* $(W, \Omega) \to (W', \Omega')$ is a homeomorphism of pairs $f: (W, \Omega) \to (W', \Omega')$ preserving all the data. More precisely, f should carry z to z', preserve the splitting of the graphs into strata and the framing, preserve the orientations of W, W' and of the strata of Ω, Ω', satisfy

$$g' \circ (f|_{C_\Omega} : C_\Omega \to C_{\Omega'})_\# = g\colon \pi_1(C_\Omega, z) \to G,$$

and satisfy $u'_{f\circ\gamma} = u_\gamma$ (resp. $v'_{f\circ\gamma} = v_\gamma$) for any path γ in C_Ω from z to a segment or a circle of $\widetilde{\Omega}$ (resp. a coupon of $\widetilde{\Omega}$). For example, if $W' = W$ and Ω' is obtained from Ω via an ambient isotopy $\{f_t : W \approx W\}_{t\in[0,1]}$ (where $f_0 = \mathrm{id}_W$), then f_1 is an e-homeomorphism $(W, \Omega) \to (W', \Omega')$.

The invariant $\tau_{\mathcal{C}}$ generalizes to extended G-manifolds (W, Ω) as follows. We use the notation of (2.5.a). Present W as the result of surgery on S^3 along a framed link ℓ.

As above, W is obtained by gluing $\#\ell$ solid tori to the exterior E of ℓ in S^3. Applying isotopy to $\Omega \subset W$ we can deform Ω into $E \subset W$. Similarly, we can push the base point z into E. Thus, we may assume that $\Omega \subset E$ and $z \in E$. The inclusion $E \subset W$ induces an epimorphism

$$\pi_1(S^3 - (\ell \cup \Omega), z) = \pi_1(E - \Omega, z) \to \pi_1(W - \Omega, z) = \pi_1(C_\Omega, z).$$

Composing with g we obtain a homomorphism, $\tilde{g}\colon \pi_1(S^3 - (\ell \cup \Omega), z) \to G$. Fix an arbitrary orientation of ℓ. The triple $(\ell \cup \Omega, z, \tilde{g})$ is a G-graph in S^3. We equip ℓ with the canonical coloring as in Section 2.2. Any path in $E \subset W$ is also a path in W so that the coloring of $\Omega \subset W$ induces a coloring of Ω as a G-graph in S^3. Consider the resulting colored G-graph $\ell_{\mathrm{can}} \cup \Omega$ in S^3 and set

$$\tau_{\mathcal{C}}(W, \Omega) = \Delta_-^{\sigma(\ell)}\, \mathcal{D}^{-\sigma(\ell)-\#\ell-1} F(\ell_{\mathrm{can}} \cup \Omega) \in K.$$

2.6 Theorem. *Let Ω be a colored G-graph in a closed connected oriented 3-manifold W. Then $\tau_{\mathcal{C}}(W, \Omega)$ does not depend on the choices made in its definition and is an e-homeomorphism invariant of (W, Ω).*

Theorem 2.6 includes Theorem 2.3 as a special case $\Omega = \emptyset$. In this case the homomorphism $g\colon \pi_1(W - \Omega, z) = \pi_1(W, z) \to G$ provides W with a structure of a G-manifold and $\tau_{\mathcal{C}}(W, \Omega) = \tau_{\mathcal{C}}(W, g)$.

The proof of Theorem 2.6 reproduces the proof of Theorem 2.3 with appropriate changes (cf. [Tu2], Section II.3).

Theorem 2.6 implies that $\tau_{\mathcal{C}}(W, \Omega)$ is invariant under ambient isotopies of Ω. In particular, $\tau_{\mathcal{C}}(W, \Omega)$ is invariant under transfers of the base point. Restricting ourselves to colored G-graphs without coupons we obtain an isotopy invariant of colored G-links in W.

If $W = S^3$, then $\tau_{\mathcal{C}}(W, \Omega) = \mathcal{D}^{-1} F(\Omega)$ (to see this, take $\ell = \emptyset$).

The invariant $\tau_{\mathcal{C}}$ satisfies the following multiplicativity law:

$$\tau_{\mathcal{C}}(W_1 \# W_2, \Omega_1 \amalg \Omega_2) = \mathcal{D}\, \tau_{\mathcal{C}}(W_1, \Omega_1)\, \tau_{\mathcal{C}}(W_2, \Omega_2), \tag{2.6.a}$$

where Ω_1, Ω_2 are colored G-graphs in closed connected oriented 3-manifolds W_1, W_2 respectively. The properties of the invariant F of colored G-graphs in S^3 established in Section VI.4.7 generalize in the obvious way to colored G-graphs in closed 3-manifolds.

Taking disjoint unions of connected extended G-manifolds we obtain non-connected extended G-manifolds. The invariant $\tau_{\mathcal{C}}$ extends to them by multiplicativity.

2.7 Remarks. 1. In the definition of an e-homeomorphism in Section 2.5, one can replace the equalities $u'_{f\circ\gamma} = u_\gamma$ with isomorphisms $u'_{f\circ\gamma} \approx u_\gamma$ in $\mathcal{C}$ and replace the equalities $v'_{f\circ\gamma} = v_\gamma$ with commutative diagrams involving these isomorphisms and the morphisms $v'_{f\circ\gamma}, v_\gamma$. We leave the details to the reader.

2. The group G acts in a canonical way on the class of 3-dimensional extended G-manifolds. The action of $\alpha \in G$ on a pair (W, Ω) with Ω as in (2.5.a) preserves

W, Ω, and the base point $z \in C_\Omega$ and replaces $(g\colon \pi_1(C_\Omega, z) \to G, u, v)$ with $(\alpha g \alpha^{-1}\colon \pi_1(C_\Omega, z) \to G, \varphi_\alpha u, \varphi_\alpha v)$, where $\varphi_\alpha u$ and $\varphi_\alpha v$ are obtained by applying φ_α to the objects and morphisms of $\mathcal{C}$ forming u and v, respectively. The same argument as in the proof of Theorem 2.3 shows that $\tau_{\mathcal{C}}$ is preserved under this action of G.

VII.3 Homotopy modular functor

3.1 Preliminaries. A 3-dimensional topological quantum field theory (TQFT) derived from a modular category has two ingredients: a modular functor assigning modules to surfaces and an operator invariant of 3-dimensional cobordisms; see [Tu2]. The surfaces must have an additional structure consisting of a finite (possibly empty) family of marked points and a Lagrangian space in real 1-dimensional homology; such surfaces are said to be *extended*. The modules associated in this theory with extended surfaces sometimes are called *modules of conformal blocks*.

Similarly, the modular crossed G-category $\mathcal{C}$ gives rise to a 3-dimensional HQFT with target $K(G, 1)$. This HQFT has two ingredients: a homotopy modular functor assigning modules to extended G-surfaces and an operator invariant of 3-dimensional G-cobordisms. In this and the next sections we discuss the homotopy modular functor. The invariant of 3-dimensional G-cobordisms derived from $\mathcal{C}$ will be discussed in Section 5. For $G = 1$, we recover the standard theory.

Recall that a topological space is pointed if all its connected components are provided with base points. In the rest of this chapter, a *map* between pointed spaces is a continuous map carrying base points to base points and considered up to homotopy constant on the base points.

3.2 Extended G-surfaces. An *extended G-surface without marks* is a pointed closed oriented surface Υ endowed with a map $\Upsilon \to X = K(G, 1)$ and with a Lagrangian space $\lambda \subset H_1(\Upsilon; \mathbb{R})$. According to our conventions, the map $\Upsilon \to X$ carries the base points of all the components of Υ into the base point $x \in X$ and is considered up to homotopy constant on the base points. Recall that a *Lagrangian space* in $H_1(\Upsilon; \mathbb{R})$ is a linear subspace of maximal dimension (equal to $\frac{1}{2} \dim H_1(\Upsilon; \mathbb{R})$) on which the homological intersection form $H_1(\Upsilon; \mathbb{R}) \times H_1(\Upsilon; \mathbb{R}) \to \mathbb{R}$ is zero.

Now we define more general extended G-surfaces with marks. Let Υ be a pointed closed oriented surface. A point $p \in \Upsilon$ is *marked* if is equipped with a sign $\varepsilon_p = \pm 1$ and a tangent direction, i.e., a ray $\mathbb{R}_+ v$, where v is a non-zero tangent vector at p. A *marking* of Υ is a finite (possibly empty) set of distinct marked points $P \subset \Upsilon$ disjoint from the base points (of the components) of Υ. Pushing slightly a marked point $p \in P$ in the given tangent direction we obtain another point $\tilde{p} \in \Upsilon$ which in analogy with knot theory can be viewed as a “longitude” of p. Set $\widetilde{P} = \bigcup_{p \in P} \tilde{p} \subset \Upsilon - P$.

Clearly, $\mathrm{card}(\widetilde{P}) = \mathrm{card}(P)$. For a path γ in $\Upsilon - P$ from a point z to $\tilde{p}$, denote by $\mu_\gamma \in \pi_1(\Upsilon - P, z)$ the homotopy class of the loop $(\gamma m_p \gamma^{-1})^{\varepsilon_p} = \gamma\, m_p^{\varepsilon_p}\, \gamma^{-1}$, where $p \in P$ and m_p is a small loop in $\Upsilon - P$ beginning and ending at $\tilde{p}$ and encircling p in the clockwise direction. (The clockwise direction is opposite to the one induced by the orientation of Υ.)

A *G-marking* of Υ is a marking $P \subset \Upsilon$ endowed with a map $g\colon \Upsilon - P \to X$ carrying the base points of Υ into $x \in X$ and considered up to homotopy constant on the base points. A G-marking $P \subset \Upsilon$ is *colored* if it is equipped with a function u which assigns to every path $\gamma\colon [0,1] \to \Upsilon - P$ from a base point, $z \in \Upsilon$, to $\widetilde{P}$ an object $u_\gamma \in \mathcal{C}$ such that

(i) u_γ is preserved under homotopies of γ in $\Upsilon - P$ fixing the endpoints;

(ii) $u_\gamma \in \mathcal{C}_{g(\mu_\gamma)}$, where by abuse of notation we denote by g the homomorphism $\pi_1(\Upsilon - P, z) \to G = \pi_1(X, x)$ induced by the mapping $g\colon \Upsilon - P \to X$;

(iii) if β is a loop in $(\Upsilon - P, z)$, then $u_{\beta\gamma} = \varphi_{g([\beta])}(u_\gamma)$, where $[\beta] \in \pi_1(\Upsilon - P, z)$ is the homotopy class of β.

An *extended G-surface* is a pointed closed oriented surface Υ with colored G-marking $P \subset \Upsilon$ and a Lagrangian space $\lambda = \lambda(\Upsilon) \subset H_1(\Upsilon; \mathbb{R})$. For $P = \emptyset$, we obtain an extended G-surface without marks as above. A disjoint union of a finite number of extended G-surfaces is an extended G-surface in the obvious way (the Lagrangian space in homology is obtained via $\oplus$). The empty set is considered as an empty extended G-surface. A *weak e-homeomorphism* of extended G-surfaces

$$f\colon (\Upsilon, P, g, u, \lambda) \to (\Upsilon', P', g', u', \lambda')$$

is formed by a homeomorphism of pairs $(\Upsilon, P) \to (\Upsilon', P')$ and a system of isomorphisms $\tilde{f}_\gamma\colon u_\gamma \to u'_{f\circ\gamma}$ in $\mathcal{C}$ labeled by paths γ in $\Upsilon - P$ from the base point of a component of Υ to $\widetilde{P}$ such that

- f preserves the orientation and the base points of the ambient surface as well as the signs of the marked points and their tangent directions;
- $g'f = g\colon \Upsilon - P \to X$ (up to homotopy constant on the base points) and for any path γ in $\Upsilon - P$ from the base point z of a component of Υ to $\widetilde{P}$, the isomorphism $\tilde{f}_\gamma\colon u_\gamma \to u'_{f\circ\gamma}$ is preserved under homotopies of γ fixing the endpoints and satisfies $\tilde{f}_{\beta\gamma} = \varphi_{g([\beta])}(\tilde{f}_\gamma)$ for any loop β in $(\Upsilon - P, z)$.

A weak e-homeomorphism f as above is an *e-homeomorphism* if the induced isomorphism $f_*\colon H_1(\Upsilon; \mathbb{R}) \to H_1(\Upsilon'; \mathbb{R})$ maps λ onto λ'.

For any extended G-surface Υ, the *opposite* extended G-surface $-\Upsilon$ is obtained from Υ by reversing the orientation of Υ and multiplying the signs of all the marked points by -1 while keeping the rest of the data. Clearly, $-(-\Upsilon) = \Upsilon$. The transformation $\Upsilon \mapsto -\Upsilon$ is natural in the sense that any (weak) e-homeomorphism $f\colon \Upsilon \to \Upsilon'$ gives rise to a (weak) e-homeomorphism $-f\colon -\Upsilon \to -\Upsilon'$ which coincides with f as a mapping.

3.3 The modular functor. The modular crossed G-category $\mathcal{C}$ gives rise to a 2-dimensional homotopy modular functor $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$. This functor assigns

- to each extended G-surface Υ a projective K-module of finite type $\mathcal{T}(\Upsilon)$;
- to each weak e-homeomorphism of extended G-surfaces $f: \Upsilon \to \Upsilon'$ an isomorphism $f_\#: \mathcal{T}(\Upsilon) \to \mathcal{T}(\Upsilon')$.

A construction of $\mathcal{T}$ will be outlined in Section 5. We state here a few simple properties of $\mathcal{T}$:

(3.3.1) $\mathcal{T}(\Upsilon \amalg \Upsilon') = \mathcal{T}(\Upsilon) \otimes_K \mathcal{T}(\Upsilon')$ for any disjoint Υ, Υ';

(3.3.2) $\mathcal{T}(\emptyset) = K$;

(3.3.3) the isomorphism $f_\#$ associated to a weak e-homeomorphism f is invariant under isotopy of f in the class of weak e-homeomorphisms;

(3.3.4) for any weak e-homeomorphisms $f: \Upsilon \to \Upsilon'$ and $f': \Upsilon' \to \Upsilon''$,

$$(f'f)_\# = (\mathcal{D}\Delta_-^{-1})^{M(f_*(\lambda(\Upsilon)),\lambda(\Upsilon'),(f')_*^{-1}(\lambda(\Upsilon'')))} f'_\# f_\#, \tag{3.3.a}$$

where f_*, f'_* denote the action of f, f' in the real 1-homology and M is the Maslov index of triples of Lagrangian subspaces of $H_1(\Upsilon'; \mathbb{R})$ (see, for instance, [Tu2], Chapter IV, for the definition and properties of the Maslov index);

(3.3.5) for any extended G-surface Υ, there is a non-degenerate bilinear pairing $\eta_\Upsilon : \mathcal{T}(\Upsilon) \otimes_K \mathcal{T}(-\Upsilon) \to K$. The pairings $\{\eta_\Upsilon\}_\Upsilon$ are natural with respect to weak e-homeomorphisms, multiplicative with respect to disjoint unions, and symmetric in the sense that $\eta_{-\Upsilon}$ is the composition of η_Υ with the standard flip $\mathcal{T}(-\Upsilon) \otimes_K \mathcal{T}(\Upsilon) \to \mathcal{T}(\Upsilon) \otimes_K \mathcal{T}(-\Upsilon)$.

The pairing η_Υ can be used to identify $(\mathcal{T}(\Upsilon))^* = \operatorname{Hom}_K(\mathcal{T}(\Upsilon), K)$ with $\mathcal{T}(-\Upsilon)$.

If two extended G-surfaces Υ and Υ' differ only by the choice of a Lagrangian space in homology, then the identity mapping $\mathrm{id}: \Upsilon \to \Upsilon'$ is a weak e-homeomorphism and therefore defines an isomorphism $\mathcal{T}(\Upsilon) \to \mathcal{T}(\Upsilon')$. This and equation (3.3.a) show that the projectivization of $\mathcal{T}(\Upsilon)$ does not depend on $\lambda(\Upsilon)$. Note that the numerical factor in (3.3.a) is equal to 1 if f or f' is an e-homeomorphism: in this case $M(f_*(\lambda(\Upsilon)), \lambda(\Upsilon'), (f')_*^{-1}(\lambda(\Upsilon''))) = 0$.

A connected extended G-surface and the corresponding module can be explicitly described (at least up to isomorphism) as follows. Let Υ be a closed connected oriented surface of genus $n \geq 0$ with base point z and marking $P = \{p_1, \dots, p_m\} \subset \Upsilon - \{z\}$, where $m \geq 0$. Let $\varepsilon_r = \pm 1$ be the sign of p_r for $r = 1, \dots, m$. For each r, choose an embedded arc γ_r in $\Upsilon - P$ leading from z to $\tilde{p}_r$. We assume that these m arcs are disjoint except at their common endpoint z. Recall from Section 3.2 the homotopy class $\mu_{\gamma_r} \in \pi_1(\Upsilon - P, z)$ of the loop encircling p_r. The group $\pi_1(\Upsilon - P, z)$ is generated by $\mu_{\gamma_1}, \dots, \mu_{\gamma_m}$ and $2n$ elements $a_1, b_1, \dots, a_n, b_n$ subject to the only relation

$$(\mu_{\gamma_1})^{\varepsilon_1} \dots (\mu_{\gamma_m})^{\varepsilon_m} [a_1, b_1] \dots [a_n, b_n] = 1, \tag{3.3.b}$$

where $[a,b] = aba^{-1}b^{-1}$. We can also assume that the homological intersection number $a_s \cdot b_s$ is $+1$ for all $s = 1, \dots, n$. Given $m + 2n$ elements $\mu_1, \dots, \mu_m, \alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ of G satisfying

$$(\mu_1)^{\varepsilon_1} \dots (\mu_m)^{\varepsilon_m} [\alpha_1, \beta_1] \dots [\alpha_n, \beta_n] = 1,$$

the formulas $\mu_{\gamma_r} \mapsto \mu_r, a_s \mapsto \alpha_s, b_s \mapsto \beta_s$ with $r = 1, \dots, m$, $s = 1, \dots, n$, define a homomorphism $\pi_1(\Upsilon - P, z) \to G$ or equivalently a map $g \colon \Upsilon - P \to X$. This turns P into a G-marking. For any objects $\{U_r \in \mathcal{C}_{\mu_r}\}_{r=1}^m$, there is a unique coloring u of P such that $u_{\gamma_r} = U_r$ for all r. The linear subspace λ of $H_1(\Upsilon; \mathbb{R})$ generated by the homology classes of $a_1, \dots, a_n$ is a Lagrangian space. The tuple $(\Upsilon, P, g, u, \lambda)$ is an extended G-surface. Recall the notation $U_r^+ = U_r, U_r^- = U_r^*$. Then

$$\begin{aligned} &\mathcal{T}(\Upsilon, P, g, u, \lambda) && \text{(3.3.c)} \\ &= \bigoplus_{i_1 \in I_{\alpha_1}, \dots, i_n \in I_{\alpha_n}} \operatorname{Hom}_{\mathcal{C}} \Big(\mathbb{1}, U_1^{\varepsilon_1} \otimes \dots \otimes U_m^{\varepsilon_m} \otimes \bigotimes_{s=1}^{n} (V_{i_s}^{\alpha_s} \otimes (\varphi_{\beta_s}(V_{i_s}^{\alpha_s}))^*)\Big), \end{aligned}$$

where for $\alpha \in G$ we denote by I_α the set of isomorphism classes of simple objects of $\mathcal{C}_\alpha$ and pick a set $\{V_i^\alpha\}_{i \in I_\alpha}$ of representatives of these classes. It is useful to note that the objects $V_i^\alpha \otimes (\varphi_\beta(V_i^\alpha))^* \in \mathcal{C}_{[\alpha,\beta]}$ determined by different representatives $V_i^\alpha = V, V'$ of $i \in I_\alpha$ are canonically isomorphic; the isomorphism in question is induced by an isomorphism $V \approx V'$ but does not depend on its choice.

It is clear that any extended G-surface Υ is weakly e-homeomorphic to an extended G-surface $(\Upsilon, P, g, u, \lambda)$ of the type described in the previous paragraph. Formula (3.3.c) allows to compute $\mathcal{T}(\Upsilon)$. Note that the group of the isotopy classes of (weak) e-homeomorphisms of Υ onto itself acts projectively on $\mathcal{T}(\Upsilon)$.

Most properties of the modular functors known for $G = 1$ extend to the general case. In particular, the splitting formula for the modules of conformal blocks along simple closed curves on surfaces extends to our setting. Instead of giving a detailed statement, we formulate the key algebraic fact underlying this formula.

3.4 Lemma. *Let $\alpha \in G$ and $\{V_i^\alpha\}_{i \in I_\alpha}$ be representatives of the isomorphism classes of simple objects in the category $\mathcal{C}_\alpha$. For any objects $V \in \mathcal{C}_\alpha$, $W \in \mathcal{C}_{\alpha^{-1}}$, there is a canonical isomorphism*

$$\bigoplus_{i \in I_\alpha} \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, V \otimes (V_i^\alpha)^*) \otimes_K \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, V_i^\alpha \otimes W) = \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, V \otimes W).$$

Proof. This is equivalent to

$$\bigoplus_{i \in I_\alpha} \operatorname{Hom}_{\mathcal{C}}(V_i^\alpha, V) \otimes_K \operatorname{Hom}_{\mathcal{C}}(W^*, V_i^\alpha) = \operatorname{Hom}_{\mathcal{C}}(W^*, V). \tag{3.4.a}$$

The latter isomorphism carries $h \otimes h'$ into $h \circ h'$ for all $h \in \operatorname{Hom}_{\mathcal{C}}(V_i^\alpha, V)$ and $h' \in \operatorname{Hom}_{\mathcal{C}}(W^*, V_i^\alpha)$. Equality (3.3.a) involves only morphisms in the category $\mathcal{C}_\alpha$ and follows from Axiom (1.1.3), cf. [Tu2], Lemma II.4.2.2. □

3.5 Action of the mapping class group. Let Υ be a pointed closed oriented surface with marking P. By a *self-homeomorphism* of (Υ, P) we shall mean an orientation preserving homeomorphism $f: \Upsilon \to \Upsilon$ keeping (set-wise) the set P and the set of base points of the components and keeping the signs and the tangent directions at the points of P. The isotopy classes of self-homeomorphisms of (Υ, P) form a group $\mathrm{Homeo}(\Upsilon, P)$ called the *mapping class group* of (Υ, P). The homotopy modular functor $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ gives rise to a projective linear action of this group as follows. Pick a Lagrangian space $\lambda \subset H_1(\Upsilon; \mathbb{R})$ and set

$$T(\Upsilon, P) = \bigoplus_{(g,u)} \mathcal{T}(\Upsilon, P, g, u, \lambda),$$

where (g, u) run over all possible pairs turning P into a colored G-marking as in Section 3.2. We define an action of a self-homeomorphism f of (Υ, P) on $T(\Upsilon, P)$. It suffices to define this action on each direct summand $\mathcal{T}(\Upsilon, P, g, u, \lambda)$. To this end, extend f to a weak e-homeomorphism $(\Upsilon, P, g, u, \lambda) \to (\Upsilon, P, g', u', \lambda)$, where $g' = gf^{-1}$, $u'_\gamma = u_{f^{-1}\gamma}$, and $\tilde{f}_\gamma = \mathrm{id}$ for all paths γ. This weak e-homeomorphism induces an isomorphism

$$\mathcal{T}(\Upsilon, P, g, u, \lambda) \to \mathcal{T}(\Upsilon, P, g', u', \lambda) \subset T(\Upsilon, P)$$

which we take as the action of f. By (3.3.3) and (3.3.4), this defines a projective linear action of $\mathrm{Homeo}(\Upsilon, P)$ on $T(\Upsilon, P)$. A different choice of λ gives a conjugate action.

If $P = \emptyset$ and the surface Υ is connected with base point z, then $\mathrm{Homeo}(\Upsilon, P) = \mathrm{Aut}(\pi_1(\Upsilon, z))$ (the group of automorphisms of $\pi_1(\Upsilon, z)$) and the module $T(\Upsilon, P)$ is a direct sum of projective K-modules of finite type determined by homomorphisms $\pi_1(\Upsilon, z) \to G$. If the group G is finite, then $T(\Upsilon, P)$ is a projective K-module of finite type. For $P \neq \emptyset$, the module $T(\Upsilon, P)$ may be very big. An action of $\mathrm{Homeo}(\Upsilon, P)$ on a smaller module can be obtained by considering only the colorings assigning simple objects to all paths γ.

3.6 Action of G. There is a canonical left action of the group G on extended G-surfaces. The action of $\alpha \in G$ transforms an extended G-surface $\Upsilon = (\Upsilon, P, g, u, \lambda)$ into ${}^\alpha\Upsilon = (\Upsilon, P, \alpha_* g, \varphi_\alpha u, \lambda)$ where $\alpha_*: (X, x) \to (X, x)$ is the map inducing the conjugation $\beta \mapsto \alpha\beta\alpha^{-1}$ in $\pi_1(X, x) = G$ and φ_α is the action of α on $\mathcal{C}$. For a (weak) e-homeomorphism of extended G-surfaces $f: \Upsilon \to \Upsilon'$, we define ${}^\alpha f$ to be the same map f viewed as a (weak) e-homeomorphism ${}^\alpha\Upsilon \to {}^\alpha\Upsilon'$.

The modular functor $\mathcal{T}$ can be enriched as follows: for every extended G-surface Υ and each $\alpha \in G$ there is a canonical isomorphism $\alpha_*: \mathcal{T}(\Upsilon) \to \mathcal{T}({}^\alpha\Upsilon)$; see Section 5.3 below. For any $\alpha, \beta \in G$, we have $(\alpha\beta)_* = \alpha_*\beta_*$. For a weak e-homeomorphism of extended G-surfaces $f: \Upsilon \to \Upsilon'$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\Upsilon) & \xrightarrow{f_\#} & \mathcal{T}(\Upsilon') \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathcal{T}({}^\alpha\Upsilon) & \xrightarrow{({}^\alpha f)_\#} & \mathcal{T}({}^\alpha\Upsilon'). \end{array}$$

If Υ is connected, then the isomorphism $\alpha_*: \mathcal{T}(\Upsilon) \to \mathcal{T}({}^\alpha\Upsilon)$ can be described in terms of $\varphi_\alpha: \mathcal{C} \to \mathcal{C}$, the splitting (3.3.c) of $\mathcal{T}(\Upsilon)$, and the similar splitting of $\mathcal{T}({}^\alpha\Upsilon)$. Namely, α_* maps the direct summand

$$\operatorname{Hom}\big(\mathbb{1}, U_1^{\varepsilon_1} \otimes \cdots \otimes U_m^{\varepsilon_m} \otimes \bigotimes_{s=1}^{n} (V_{i_s}^{\alpha_s} \otimes (\varphi_{\beta_s}(V_{i_s}^{\alpha_s}))^*)\big)$$

via φ_α on the direct summand

$$\operatorname{Hom}\big(\mathbb{1}, (\varphi_\alpha U_1)^{\varepsilon_1} \otimes \cdots \otimes (\varphi_\alpha U_m)^{\varepsilon_m} \otimes \bigotimes_{s=1}^{n} (V_{\varphi_\alpha(i_s)} \otimes (\varphi_{\alpha\beta_s\alpha^{-1}}(V_{\varphi_\alpha(i_s)}))^*)\big)$$

of $\mathcal{T}({}^\alpha\Upsilon)$ corresponding to the tuple $\varphi_\alpha(i_1) \in I_{\alpha\alpha_1\alpha^{-1}}, \dots, \varphi_\alpha(i_n) \in I_{\alpha\alpha_n\alpha^{-1}}$. Here $V_{\varphi_\alpha(i_s)} = V_{\varphi_\alpha(i_s)}^{\alpha\alpha_s\alpha^{-1}}$ is a simple object in $\mathcal{C}_{\alpha\alpha_s\alpha^{-1}}$ representing the isomorphism class $\varphi_\alpha(i_s)$ of $\varphi_\alpha(V_{i_s})$. For instance, we can take $V_{\varphi_\alpha(i_s)} = \varphi_\alpha(V_{i_s})$.

The isomorphism $\alpha_*: \mathcal{T}(\Upsilon) \to \mathcal{T}({}^\alpha\Upsilon)$ can be sometimes computed in terms of the action of homeomorphisms. Assume for simplicity that $\Upsilon = (\Upsilon, P, g, u, \lambda)$ is connected and $z \in \Upsilon - P$ is its base point. Given a loop γ in $\Upsilon - P$ based at z, consider an isotopy $\mathrm{id}_\Upsilon \sim f$ of the identity map pushing z along γ and constant on P. Then f is an e-homeomorphism $\Upsilon \to {}^\alpha\Upsilon$, where $\alpha = [g \circ \gamma] \in G$ and $\alpha_* = f_\#: \mathcal{T}(\Upsilon) \to \mathcal{T}({}^\alpha\Upsilon)$.

3.7 Computations on the torus. Consider the case where Υ is a torus without marks. The group $\pi_1(\Upsilon, z)$ is generated by two commuting elements a, b. A map $g: \Upsilon \to X$ is determined by two commuting elements $\alpha, \beta \in G$, the images of a and b, respectively. The 1-dimensional subspace λ of $H_1(\Upsilon; \mathbb{R})$ generated by the homology class of a is a Lagrangian space. By (3.3.c),

$$\mathcal{T}(\Upsilon, g, \lambda) = \bigotimes_{i \in I_\alpha} \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, V_i^\alpha \otimes (\varphi_\beta(V_i^\alpha))^*) = \bigoplus_{i \in I_\alpha} \operatorname{Hom}_{\mathcal{C}}(\varphi_\beta(V_i^\alpha), V_i^\alpha).$$

Observe that for any simple objects U, U' of $\mathcal{C}$,

$$\operatorname{Hom}_{\mathcal{C}}(U, U') = \begin{cases} K & \text{if } U \text{ is isomorphic to } U', \\ 0 & \text{otherwise.} \end{cases}$$

Note also that φ_β maps $\mathcal{C}_\alpha$ into itself and induces a permutation on the set I_α. Therefore $\mathcal{T}(\Upsilon, g, \lambda) = K^N$, where $N = \operatorname{card}\{i \in I_\alpha \mid \varphi_\beta(i) = i\}$. For instance, if $\beta = 1$, then $N = \mathrm{card}(I_\alpha)$. In this case we can use Lemma 1.2 to identify $T(\Upsilon, g, \lambda)$ with the module L_α. More generally, if the image of the homomorphism $\pi_1(\Upsilon) \to G$ induced by g is a cyclic group generated by $\alpha \in G$, then $\mathcal{T}(\Upsilon, g, \lambda) = K^{\mathrm{card}(I_\alpha)}$. Indeed, we can choose generators $a, b \in \pi_1(\Upsilon)$ as above so that their images in G are equal to α and 1, respectively.

Consider the case where Υ is a torus with one marked point p, i.e., $m = n = 1$ in the notation of Section 3.3. We shall omit the index 1 and write p for p_1, ε for ε_1, etc. Thus $P = \{p\}$ and the group $\pi_1(\Upsilon - P, z)$ is generated by three elements μ_γ, a, b subject to the relation $(\mu_\gamma)^\varepsilon [a, b] = 1$. The map $g \colon \Upsilon - P \to X$ is determined by three elements μ, α, β of G, the images of μ_γ, a, b, respectively. They must satisfy the equality $\mu^\varepsilon [\alpha, \beta] = 1$. The coloring u of P is determined by $u_\gamma = U \in \mathcal{C}_\mu$. The 1-dimensional subspace λ of $H_1(\Upsilon; \mathbb{R})$ generated by the homology class of a is a Lagrangian space. Then

$$\mathcal{T}(\Upsilon, P, g, u, \lambda) = \bigoplus_{i \in I_\alpha} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, U^\varepsilon \otimes V_i^\alpha \otimes (\varphi_\beta(V_i^\alpha))^*).$$

Taking different systems (3.3.b) of generators of $\pi_1(\Upsilon - P, z)$, we obtain different expansions (3.3.c) of the module $\mathcal{T}(\Upsilon, P, g, u, \lambda)$. This gives non-trivial identities between the Hom-spaces in $\mathcal{C}$. For instance, in the case $m = n = 1$, the identity $[a, b] = [aba^{-1}, a^{-1}]$ implies that the group $\pi_1(\Upsilon - P, z)$ is generated by three elements $\mu_\gamma, aba^{-1}, a^{-1}$ subject to the relation $(\mu_\gamma)^\varepsilon [aba^{-1}, a^{-1}] = 1$. The same map $g \colon \Upsilon - P \to X$ as above carries these generators to $\mu, \alpha\beta\alpha^{-1}, \alpha^{-1}$, respectively. The same coloring u is determined by $u_\gamma = U$. Thus, under these two choices of generators of $\pi_1(\Upsilon - P, z)$ we obtain two extended G-surfaces which differ only in the Lagrangian spaces generated by the homology classes of a and b, respectively. The associated K-modules are therefore isomorphic:

$$\begin{aligned} &\bigoplus_{i \in I_\alpha} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, U^\varepsilon \otimes V_i^\alpha \otimes (\varphi_\beta(V_i^\alpha))^*) \\ &\quad = \bigoplus_{j \in I_{\alpha\beta\alpha^{-1}}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, U^\varepsilon \otimes V_j^{\alpha\beta\alpha^{-1}} \otimes (\varphi_{\alpha^{-1}}(V_j^{\alpha\beta\alpha^{-1}}))^*) \\ &\quad = \bigoplus_{k \in I_\beta} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, U^\varepsilon \otimes \varphi_\alpha(V_k^\beta) \otimes (V_k^\beta)^*), \end{aligned} \tag{3.7.a}$$

where the last equality follows from the fact that φ_α induces a bijection $I_\beta \to I_{\alpha\beta\alpha^{-1}}$.

VII.4 Two-dimensional HQFT

The homotopy modular functor $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ associated with a modular crossed G-category $\mathcal{C}$ naturally gives rise to an "underlying" 2-dimensional HQFT (A, τ) with target $X = K(G, 1)$. We outline the construction and properties of (A, τ). This HQFT can be alternatively obtained from the 3-dimensional HQFT derived from $\mathcal{C}$ in the next section: one should multiply the 1-dimensional X-manifolds and 2-dimensional X-cobordisms by S^1, and apply the 3-dimensional HQFT.

4.1 Construction of (A, τ). We shall use the terminology introduced in Section 1.1 of Chapter III. For a connected X-curve M, set $A_M = L_\alpha$, where $\alpha \in G$ is the homotopy class of the loop $M \to X$ and L_α is the K-module defined in Section VI.5.3. Then extend A to non-connected M by multiplicativity as in Section III.1.1. The value of τ on an X-surface $(W, M_0, M_1, g\colon W \to X)$ is a K-homomorphism

$$\tau(W, g)\colon A_{M_0} = \bigotimes_{c \subset M_0} L_{\alpha_c} \to A_{M_1} = \bigotimes_{c \subset M_1} L_{\alpha_c},$$

where c runs over the components of M_0 (resp. of M_1) and $\alpha_c \in G$ is the homotopy class of the loop $g|_c$ in (X, x). The homomorphism $\tau(W, g)$ is obtained, roughly speaking, by taking the dimension of the module assigned by $\mathcal{T}$ to an extended G-surface associated with W. In particular, if $M_0 = M_1 = \emptyset$, then $\tau(W, g) = \operatorname{Dim} \mathcal{T}(\Upsilon_{W,g}) \in K$, where $\Upsilon_{W,g}$ is an extended G-surface without marks obtained from W by choosing an arbitrary Lagrangian space in $H_1(W; \mathbb{R})$, arbitrary base points on the components of W and deforming g so that it maps these base points into x. By Section 3, the isomorphism class of the module $\mathcal{T}(\Upsilon_{W,g})$ and therefore its dimension Dim do not depend on the choices made in the construction of $\Upsilon_{W,g}$.

To define $\tau(W, g)$ for an arbitrary X-surface $(W, M_0, M_1, g\colon W \to X)$, we proceed as follows. For each component c of $\partial W = M_0 \cup M_1$, the K-module L_{α_c} is free with basis given by Lemma 1.2. We present $\tau(W, g)$ by a matrix with respect to the tensor products of these bases. Pick for each component c of ∂W a simple object $V_c \in \mathcal{C}_{\alpha_c}$. Consider the basis elements

$$\bigotimes_{c \subset M_0} \langle V_c \rangle \in \bigotimes_{c \subset M_0} L_{\alpha_c} \quad \text{and} \quad \bigotimes_{c \subset M_1} \langle V_c \rangle \in \bigotimes_{c \subset M_1} L_{\alpha_c}.$$

The corresponding matrix term $\langle \tau(W, g) \,|\, \{V_c\}_c \rangle$ of $\tau(W, g)$ is defined as follows. We first upgrade W to an extended G-surface with marks. Assume for simplicity that W is connected and fix a base point $z \in W - \partial W$. We deform the mapping $g\colon W \to X$ rel ∂W so that $g(z) = x$. We cap W with 2-discs by gluing to each component c of ∂W a copy of the unit complex 2-disc $D = \{a \in \mathbb{C} \mid |a| \leq 1\}$. The gluing is done so that the point $1 \in \partial D$ is identified with the base point of c. This gives a closed surface Υ which we provide with orientation extending the one in W. The centers (corresponding to $0 \in D$) of the glued 2-discs form a finite set $P \subset \Upsilon$. We provide each $p \in P$ with tangent direction corresponding to $\mathbb{R}_+ \subset \mathbb{C}$. Pushing $p \in P$ along this tangent direction we obtain the base point $\tilde{p}$ of the corresponding component, c_p, of ∂W. Set $\varepsilon_p = +1$ if $c_p \subset M_1$ and $\varepsilon_p = -1$ if $c_p \subset M_0$.

Clearly, W is a deformation retract of $\Upsilon - P$. Therefore the map $g\colon W \to X$ extends to a map $\Upsilon - P \to X$ also denoted g. This turns P into a G-marking on Υ.

We color P as follows. For $p \in P$ and a path $\gamma\colon [0,1] \to \Upsilon - P$ from z to $\tilde{p}$, the path $g \circ \gamma$ is a *loop* in (X, x) so that we can consider its homotopy class $[g \circ \gamma] \in G$. Set

$$u_\gamma = \begin{cases} \varphi_{[g\circ\gamma]}(V_{c_p}) & \text{if } \varepsilon_p = +1, \\ \varphi_{[g\circ\gamma]}(V_{c_p}^*) & \text{if } \varepsilon_p = -1. \end{cases}$$

Conditions (i)–(iii) of Section 3.2, p. 171, are straightforward. In particular, Condition (ii) follows from the equality

$$g(\mu_\gamma) = [g \circ \gamma](\alpha_{c_p})^{\varepsilon_p}[g \circ \gamma]^{-1}.$$

Choosing an arbitrary Lagrangian space $\lambda \subset H_1(\Upsilon; \mathbb{R})$, we obtain an extended G-surface $(\Upsilon, g, P, u, \lambda)$. Set

$$\langle \tau(W, g) \,|\, \{V_c\}_c \rangle = \operatorname{Dim} \mathcal{T}(\Upsilon, g, P, u, \lambda) \in K.$$

One can check that (A, τ) is a 2-dimensional X-HQFT. We shall not do it here but rather describe the structure of a crossed G-algebra in $L = \bigoplus_{\alpha\in G} L_\alpha$ underlying this HQFT. First, define a pairing $\eta\colon L \times L \to K$ by

$$\eta(\langle U, f\rangle, \langle U', f'\rangle) = \operatorname{Tr}(f \otimes f')_*,$$

where $\langle U, f\rangle, \langle U', f'\rangle$ are additive generators of L as in Section VI.5.3 and $(f \otimes f')_*$ denotes the endomorphism of $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, U \otimes U')$ carrying each $h \in \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, U \otimes U')$ to $(f \otimes f')h$. It is easy to check that η is a well-defined bilinear pairing.

4.2 Theorem. *The G-graded algebra $L = \bigoplus_{\alpha\in G} L_\alpha$ with the form η and the action φ of G defined in Section* VI.5.3 *is a crossed G-algebra.*

Proof. For each $\alpha \in G$, fix representatives $\{V_i^\alpha\}_{i\in I_\alpha}$ of the isomorphism classes of simple objects in the category $\mathcal{C}_\alpha$. By Lemma 1.2, L_α is a free K-module with basis $\{\langle V_i^\alpha\rangle\}_{i\in I_\alpha}$. Observe that for any objects U, U' of $\mathcal{C}$,

$$\eta(\langle U\rangle, \langle U'\rangle) = \operatorname{Dim}(\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, U \otimes U')) = \operatorname{Dim}(\operatorname{Hom}_{\mathcal{C}}(U^*, U')).$$

In particular, if U, U' are simple, then

$$\eta(\langle U\rangle, \langle U'\rangle) = \begin{cases} 1 & \text{if } U' \text{ is isomorphic to } U^*, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.a}$$

Hence $\eta(L_\alpha \otimes L_\beta) = 0$ if $\alpha\beta \neq 1$ and the bases $\{\langle V_i^\alpha\rangle\}_{i\in I_\alpha}$ and $\{\langle V_j^{\alpha^{-1}}\rangle\}_{j\in I_{\alpha^{-1}}}$ are dual with respect to η. Therefore η is non-degenerate. Formula $\eta(ab, c) = \eta(a, bc)$ follows from the definitions. Formula (4.2.a) and the fact that U' is isomorphic to U^* if and only if U is isomorphic to $(U')^*$ (Corollary VI.4.6) imply that η is symmetric. Therefore η is an inner product on L in the sense of Section II.2.1.

Formula (4.2.a) directly implies that η is invariant under the action of G. It remains to verify Axioms (3.1.1)–(3.1.4), p. 25. The first three axioms were already checked in Section VI.5.3. It suffices to verify Axiom (3.1.4) for $c = \langle U \rangle$, where U is any object of $\mathcal{C}_{\alpha\beta\alpha^{-1}\beta^{-1}}$. By (1.2.a) the homomorphism $\mu_c\, \varphi_\beta \colon L_\alpha \to L_\alpha$ carries $\langle V_i^\alpha \rangle$ to

$$\langle U \otimes \varphi_\beta(V_i^\alpha) \rangle = \sum_{j \in I_\alpha} \nu_{V_j^\alpha, U \otimes \varphi_\beta(V_i^\alpha)} \langle V_j^\alpha \rangle.$$

Therefore

$$\begin{aligned} \operatorname{Tr}(\mu_c\, \varphi_\beta) &= \sum_{i \in I_\alpha} \nu_{V_i^\alpha, U \otimes \varphi_\beta(V_i^\alpha)} \\ &= \sum_{i \in I_\alpha} \nu_{U \otimes \varphi_\beta(V_i^\alpha), V_i^\alpha} \\ &= \sum_{i \in I_\alpha} \operatorname{Dim}(\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, U^* \otimes V_i^\alpha \otimes (\varphi_\beta(V_i^\alpha))^*)), \end{aligned} \tag{4.2.b}$$

where we use the equalities

$$\nu_{U \otimes V, W} = \nu_{U, W \otimes V^*} = \nu_{\mathbb{1}, U^* \otimes W \otimes V^*}.$$

A similar computation shows that

$$\operatorname{Tr}(\varphi_{\alpha^{-1}}\mu_c \colon L_\beta \to L_\beta) = \sum_{k \in I_\beta} \operatorname{Dim}(\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, U^* \otimes \varphi_\alpha(V_k^\beta) \otimes (V_k^\beta)^*)).$$

By (3.7.a) with $\varepsilon = -$, the right-hand sides of the latter formula and of (4.2.b) are the dimensions of isomorphic K-modules. Therefore they are equal. □

4.3 Remarks. 1. The proof of Axiom (3.1.4) in Theorem 4.2 uses the homotopy modular functor $\mathcal{T}_{\mathcal{C}}$. It would be useful to have a direct algebraic proof.

2. The neutral component L_1 of the algebra $L = L(\mathcal{C})$ is known to be a direct sum of copies of K (as an algebra), cf. [Tu2], Section IV.12.4. Therefore L is semisimple in the sense of Section II.5.1. By Corollary III.3.4 if K is a field of characteristic 0, then the HQFT (A, τ) is semi-cohomological. Thus, this HQFT is determined by a finite family of K-valued weights and 2-dimensional cohomology classes of subgroups of G of finite index.

VII.5 Three-dimensional HQFT

Each modular crossed G-category $\mathcal{C}$ gives rise to a 3-dimensional HQFT with target $X = K(G, 1)$. As in the standard case $G = 1$, this HQFT has several equivalent versions. We describe here one of these versions formulated in terms of extended (3-dimensional) G-manifolds.

5.1 Extended G-manifolds with boundary. Let W be a compact connected oriented 3-manifold with non-void pointed boundary. A *ribbon graph* $\Omega \subset W$ consists of a finite family of framed oriented embedded segments, circles and coupons (the *strata* of Ω) such that

(i) the strata of Ω are disjoint except that some endpoints of the segments lie on the bases of the coupons;

(ii) all other endpoints of the segments of Ω lie on ∂W and form the (finite) set $\Omega \cap \partial W$; this set is disjoint from the set of base points of (the components of) ∂W;

(iii) the framings of the strata form a continuous nonsingular vector field (a framing) on Ω transversal to Ω and tangent to ∂W at $\Omega \cap \partial W$;

(iv) the framing and the orientation of each coupon of Ω determine the given orientation of W.

Slightly pushing Ω along the framing, we obtain a disjoint copy $\widetilde{\Omega}$ of Ω. Pushing a stratum t of Ω along the framing, we obtain a stratum $\tilde{t}$ of $\widetilde{\Omega}$.

A *G-graph* in the manifold W is a ribbon graph $\Omega \subset W$ endowed with a mapping $g\colon C_\Omega = W - \Omega \to X$ which carries the base points of ∂W to the base point x of X. We consider g up to homotopy constant on the base points of ∂W.

Let (Ω, g) be a G-graph in W and let z be the base point of a component of ∂W. A *coloring of* Ω *with respect to* z is a pair (u, v) defined exactly as in Section VI.4.1. In this definition and below, the letter g is used both for the mapping $g\colon C_\Omega \to X$ and for the induced homomorphism $\pi_1(C_\Omega, z) \to \pi_1(X, x) = G$. A coloring (u, v) of Ω with respect to z induces a coloring (u', v') of Ω with respect to the base point, z', of any other component of ∂W. (Here it is essential that W is connected). Namely, pick any path ρ in C_Ω from z to z'. For a segment or circle t of Ω and a path γ in C_Ω from z' to $\tilde{t}$, set

$$u'_\gamma = (\varphi_{[g\circ\rho]})^{-1}(u_{\rho\gamma}).$$

Note that the path $g \circ \rho$ is a *loop* in (X, x) so that we can consider its homotopy class $[g \circ \rho] \in G$. The path $\rho\gamma$ connects z to $\tilde{t}$ so that $u_{\rho\gamma} \in \mathcal{C}_{g(\mu_{\rho\gamma})}$. We have $\mu_{\rho\gamma} = \rho\mu_\gamma\rho^{-1}$ and $g(\mu_{\rho\gamma}) = [g \circ \rho]\, g(\mu_\gamma)\, [g \circ \rho]^{-1}$. The functor $(\varphi_{[g\circ\rho]})^{-1}$ maps $\mathcal{C}_{g(\mu_{\rho\gamma})}$ to $\mathcal{C}_{g(\mu_\gamma)}$, so that u'_γ is an object of $\mathcal{C}_{g(\mu_\gamma)}$. Similarly, for a coupon Q of Ω and a path γ in C_Ω connecting z' to $\widetilde{Q}$, set

$$v'_\gamma = (\varphi_{[g\circ\rho]})^{-1}(v_{\rho\gamma}).$$

It is easy to check that (u', v') is a coloring of Ω with respect to z' independent of the choice of ρ. In this way, a coloring of Ω with respect to z canonically extends to a system of colorings of Ω with respect to the base points of all components of ∂W. This system is called a *coloring* of Ω. To specify a coloring of Ω it is enough to specify a coloring of Ω with respect to a single base point of ∂W.

A tuple $(W, \Omega, g, u, v, \lambda)$ consisting of a 3-manifold W as above, a colored G-graph (Ω, g, u, v) in W and a Lagrangian space $\lambda \subset H_1(\partial W; \mathbb{R})$ is called a *connected* *extended G-manifold with boundary*. For brevity, we shall sometimes denote such a tuple simply by W. Given a connected extended G-manifold with boundary $W = (W, \Omega, g, u, v, \lambda)$, the surface ∂W becomes an extended G-surface as follows. By definition, ∂W is pointed. We provide ∂W with the orientation induced by the one in W (see Section I.1.1 for our orientation conventions). Set $P = \Omega \cap \partial W$. Clearly, P is a finite subset of ∂W disjoint from the base points. Each point $p \in P$ is provided with the sign $-$ if the adjacent segment of Ω is oriented towards p and with the sign $+$ otherwise. We provide p with the tangent direction generated by the framing vector at p (this vector is tangent to ∂W). We can assume that $\widetilde{P} = \widetilde{\Omega} \cap \partial W$. Denote by ∂g the restriction of the map $g\colon C_\Omega \to X$ to $\partial W - P$. The pair $(P, \partial g)$ is a G-marking on ∂W. We define its coloring ∂u as follows. Every path γ in $\partial W - P$ from a base point of ∂W to $\widetilde{P}$ can be viewed a path in C_Ω leading to $\widetilde{\Omega}$. Set $(\partial u)_\gamma = u_\gamma \in \mathcal{C}$. It is clear that the tuple $(\partial W, P, \partial g, \partial u, \lambda)$ is an extended G-surface. It is called *the boundary* of $(W, \Omega, g, u, v, \lambda)$.

A *weak e-homeomorphism*

$$(W, \Omega, g, u, v, \lambda) \to (W', \Omega', g', u', v', \lambda') \tag{5.1.a}$$

of connected extended G-manifolds with boundary is a homeomorphism of pairs $f\colon (W, \Omega) \to (W', \Omega')$ such that

- f preserves the orientation in W, W' and the framing, the orientation, and the splitting of Ω, Ω' into strata (the framings on Ω, Ω' are considered up to homotopy);
- $g'f = g\colon C_\Omega \to X$ and for any path γ in C_Ω from a base point of ∂W to a segment or a circle of $\widetilde{\Omega}$ (resp. a coupon of $\widetilde{\Omega}$) we have $u'_{f\circ\gamma} = u_\gamma$ (resp. $v'_{f\circ\gamma} = v_\gamma$).

The weak e-homeomorphism (5.1.a) is an *e-homeomorphism* if the induced isomorphism $H_1(\partial W; \mathbb{R}) \to H_1(\partial W'; \mathbb{R})$ maps λ onto λ'. It is clear that a (weak) e-homeomorphism of extended G-manifolds induces a (weak) e-homeomorphism of their boundaries.

Taking disjoint unions of connected extended G-manifolds with or without boundary we obtain arbitrary extended G-manifolds. The notions of boundary and (weak) e-homeomorphisms generalize to them in the obvious way.

5.2 Extended G-cobordisms. An *extended G-cobordism* is an arbitrary triple $(W, \partial_- W, \partial_+ W)$, where W is an extended (3-dimensional) G-manifold and $\partial_- W$, $\partial_+ W$ are extended G-surfaces such that $\partial W = (-\partial_- W) \amalg \partial_+ W$. We call $\partial_- W$ and $\partial_+ W$ the *bottom base* and the *top base* of W, respectively. For instance, any extended G-manifold W gives rise to an extended G-cobordism $(W, \emptyset, \partial W)$.

An *e-homeomorphism* $(W, \partial_- W, \partial_+ W) \to (W', \partial_- W', \partial_+ W')$ of extended G-cobordisms is an e-homeomorphism $W \to W'$ carrying $\partial_\pm W$ onto $\partial_\pm W'$.

Extended G-cobordisms can be glued as follows. Let $(W_r, \partial_- W_r, \partial_+ W_r)$ be an extended G-cobordism for $r = 1, 2$ and let $f: \partial_+ W_1 \to \partial_- W_2$ be an e-homeomorphism of extended G-surfaces. We can glue W_1 and W_2 along f into a cobordism $(W, \partial_- W, \partial_+ W)$ between $\partial_- W = \partial_- W_1$ and $\partial_+ W = \partial_+ W_2$. The orientations of W_1 and W_2 extend to an orientation of W. The given ribbon graphs $\Omega_1 \subset W_1$ and $\Omega_2 \subset W_2$ are glued along $f: \Omega_1 \cap \partial_+ W_1 \to \Omega_2 \cap \partial_- W_2$ into a ribbon graph $\Omega \subset W$. Set $C = \partial_+ W_1 - (\partial_+ W_1 \cap \Omega_1)$. By the assumptions on f, the given maps $g_1: W_1 - \Omega_1 \to X$ and $g_2: W_2 - \Omega_2 \to X$ satisfy $g_2 f|_C = g_1|_C$ up to homotopy in the class of maps $C \to X$ carrying the base points of $\partial_+ W_1$ to the base point x of X. Deforming if necessary g_1, we can assume that $g_2 f|_C = g_1|_C$ as maps so that g_1, g_2 can be glued into a map $g: C_\Omega = W - \Omega \to X$. If $\partial W \neq \emptyset$, then (Ω, g) is a G-graph. If $\partial W = \emptyset$, then we choose the base point z of an arbitrary component of $\partial_+ W_1 \subset \operatorname{Int} W$ to be the base point of C_Ω and endow Ω with the homomorphism $\pi_1(C_\Omega, z) \to G$ induced by g. The pair (Ω, g) is a G-graph. The given colorings (u_1, v_1) of Ω_1 and (u_2, v_2) of Ω_2 induce a coloring (u, v) of Ω as follows. Suppose first that $\partial W \neq \emptyset$. Consider a path γ from a base point of ∂W to the parallel $\tilde{t}$ of a 1-dimensional stratum t of Ω. Deforming if necessary γ rel the endpoints, we can expand $\gamma = \gamma_1 \gamma_2 \dots \gamma_n \gamma'$, where each γ_i is a path in W_1 or in W_2 connecting the base points of certain components of the boundary and γ' is a path in W_r with $r \in \{1, 2\}$ leading from a base point of ∂W_r to $\tilde{t}$. For $i = 1, \dots, n$, the path $g\gamma_i$ is a loop in X representing an element $[g\gamma_i]$ of G. The coloring u_r of Ω_r yields an object $(u_r)_{\gamma'}$ of $\mathcal{C}$. Set

$$u_\gamma = \varphi_{[g\gamma_1]} \varphi_{[g\gamma_2]} \dots \varphi_{[g\gamma_n]}((u_r)_{\gamma'}).$$

The function v and the case $\partial W = \emptyset$ are treated similarly.

Examples of extended G-cobordisms are provided by cylinders over extended G-surfaces. Consider an extended G-surface $\Upsilon = (\Upsilon, P \subset \Upsilon, g: \Upsilon - P \to X, u, \lambda)$. Endow $\Upsilon \times [0, 1]$ with the product orientation, where the interval $[0, 1]$ is oriented from left to right. Endow $\partial(\Upsilon \times [0, 1]) = \Upsilon \times \{0, 1\}$ with base points $z \times \{0\}$ and $z \times \{1\}$, where $z \in \Upsilon - P$ runs over the base points of the components of Υ. Set $\Omega = P \times [0, 1]$ and provide each segment $p \times [0, 1]$ (where $p \in P$) with constant framing determined by a vector representing the given tangent direction at p. We orient $p \times [0, 1]$ towards $p \times \{0\}$ if the sign of p is $+$ and towards $p \times \{1\}$ otherwise. Let pr be the projection $(\Upsilon \times [0, 1]) - \Omega \to \Upsilon - P$. The map $g \circ \mathrm{pr}$ turns Ω into a G-graph and the formula $\gamma \mapsto u_{\mathrm{pr} \circ \gamma}$ defines its coloring. Finally, we provide $\Upsilon \times [0, 1]$ with the Lagrangian space in real 1-homology of the boundary equal to the direct sum of the copies of λ in $H_1(\Upsilon \times 0; \mathbb{R})$ and in $H_1(\Upsilon \times 1; \mathbb{R})$. This turns the triple $(\Upsilon \times [0, 1], \Upsilon \times 0, \Upsilon \times 1)$ into an extended G-cobordism whose bases are copies of Υ. More generally, for $\alpha \in G$, we can define a twisted cylinder $(\Upsilon \times [0, 1])^\alpha$. It differs from $\Upsilon \times [0, 1]$ only by the choice of the map $(\Upsilon \times [0, 1]) - \Omega \to X$ and the coloring of $\Omega = P \times [0, 1]$. The map in question is chosen so that its restriction to the bottom base is g and its restrictions to segments $z \times [0, 1]$ are loops representing α^{-1} for all base points z of Υ. The coloring of Ω is chosen so that its restriction to the bottom base is equal to u. Then $(\Upsilon \times [0, 1])^\alpha$

is an extended G-cobordism with bottom base Υ and top base ${}^{\alpha}\Upsilon$.

5.3 The HQFT $(\mathcal{T}_{\mathcal{C}}, \tau_{\mathcal{C}})$. The HQFT $(\mathcal{T}_{\mathcal{C}}, \tau_{\mathcal{C}})$ derived from a modular crossed G-category $\mathcal{C}$ comprises the 2-dimensional homotopy modular functor $\mathcal{T}_{\mathcal{C}}$ discussed in Section 3 and a function $\tau_{\mathcal{C}}$ assigning to each extended 3-dimensional G-cobordism $(W, \partial_- W, \partial_+ W)$ a homomorphism

$$\tau_{\mathcal{C}}(W, \partial_- W, \partial_+ W) \colon \mathcal{T}_{\mathcal{C}}(\partial_- W) \to \mathcal{T}_{\mathcal{C}}(\partial_+ W).$$

This homomorphism is multiplicative with respect to disjoint unions of extended G-cobordisms and natural with respect to e-homeomorphisms. In the case where $\partial_- W = \partial_+ W = \emptyset$, the homomorphism $\tau_{\mathcal{C}}(W, \partial_- W, \partial_+ W) \colon K \to K$ is multiplication by the invariant $\tau_{\mathcal{C}}(W)$ introduced in Section 2. If an extended G-cobordism $(W, \partial_- W, \partial_+ W)$ is obtained by gluing extended G-cobordisms $(W_1, \partial_- W_1, \partial_+ W_1)$ and $(W_2, \partial_- W_2, \partial_+ W_2)$ along an e-homeomorphism $f \colon \partial_+ W_1 \to \partial_- W_2$, then

$$\begin{aligned}\tau_{\mathcal{C}}(W, \partial_- W, \partial_+ W) = (\mathcal{D}\Delta_-^{-1})^M\, \tau_{\mathcal{C}}(W_2, \partial_-(W_2), \partial_+(W_2)) \circ \\ \circ f_\# \circ \tau_{\mathcal{C}}(W_1, \partial_-(W_1), \partial_+(W_1)),\end{aligned}$$

where $M \in \mathbb{Z}$ is the Maslov index determined by the given Lagrangian spaces in H_1 of the bases as in [Tu2] (the number M does not depend on the maps to X). Finally, for any extended G-surface Υ, the endomorphism $\tau_{\mathcal{C}}(\Upsilon \times [0,1], \Upsilon \times 0, \Upsilon \times 1)$ of $\mathcal{T}_{\mathcal{C}}(\Upsilon)$ is the identity. More generally, for any such Υ and any $\alpha \in G$, the operator invariant of the twisted cylinder $(\Upsilon \times [0,1])^\alpha$ is nothing but the action of α discussed in Section 3.6:

$$\tau_{\mathcal{C}}((\Upsilon \times [0,1])^\alpha, \Upsilon, {}^{\alpha}\Upsilon) = \alpha_* \colon \mathcal{T}_{\mathcal{C}}(\Upsilon) \to \mathcal{T}_{\mathcal{C}}({}^{\alpha}\Upsilon).$$

The construction of $(\mathcal{T}_{\mathcal{C}}, \tau_{\mathcal{C}})$ closely follows the construction of 3-dimensional TQFTs from modular categories given in [Tu2] (the case $G = \{1\}$). First, one defines the operator invariant for extended G-cobordisms whose boundary components are parametrized, i.e., identified with standard G-surfaces in S^3. (Only the geometric position of the surfaces in S^3 is standard, the maps to X are arbitrary.) Second, one uses these operators to define the action of weak e-homeomorphisms. Third, one replaces the parametrizations with Lagrangian spaces in homology. We skip the details and give here only one of the key lemmas whose proof is somewhat different from the one in the case $G = \{1\}$.

5.4 Lemma. *Let $T \subset \mathbb{R}^2 \times [0,1]$ be a tangle formed by a vertical interval t oriented downward and its meridian m, both with zero framing. Sending all meridians of t to $1 \in G$ and all meridians of m to $\alpha \in G$, we turn T into a G-tangle T_α. Let $V \in \mathcal{C}_1$ be a simple object in the neutral component of $\mathcal{C}$. Let $T_\alpha(V)$ be T_α colored so that m acquires the canonical color as in Section 2.2 and the target of T_α is the* *triple* $(+1, 1 \in G, V)$*, see Figure* VII.1*. Then the source of* $T_\alpha(V)$ *is the triple* $(+1, 1 \in G, \varphi_{\alpha^{-1}}(V))$ *and* $F(T_\alpha(V)) \in \mathrm{Hom}_{\mathcal{C}}(\varphi_{\alpha^{-1}}(V), V)$ *is computed by*

$$F(T_\alpha(V)) = \begin{cases} \mathcal{D}^2 \,\mathrm{id}_{\mathbb{1}} & \text{if } V \text{ is isomorphic to } \mathbb{1} \text{ and } \mathcal{C}_\alpha \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\mathcal{C}_\alpha = \emptyset$, then the canonical color of m is 0 and the claim is obvious. Suppose from now on that $\mathcal{C}_\alpha \neq \emptyset$. Pick an arbitrary object W of $\mathcal{C}$. It necessarily belongs to $\mathcal{C}_\beta$ for some $\beta \in G$. Consider the G-tangle T_β as in the statement of the lemma with α replaced by β. We present T_β by a plane diagram with two crossings. We attach W to the arc of the diagram representing m and attach V (resp. $\varphi_{\beta^{-1}}(V)$) to the arc incident to the output (resp. input). Denote the resulting colored G-tangle by T_W^V, see Figure VII.1. Its target is the triple $(+1, 1 \in G, V)$, and its source is the triple $(+1, 1 \in G, \varphi_{\beta^{-1}}(V))$.

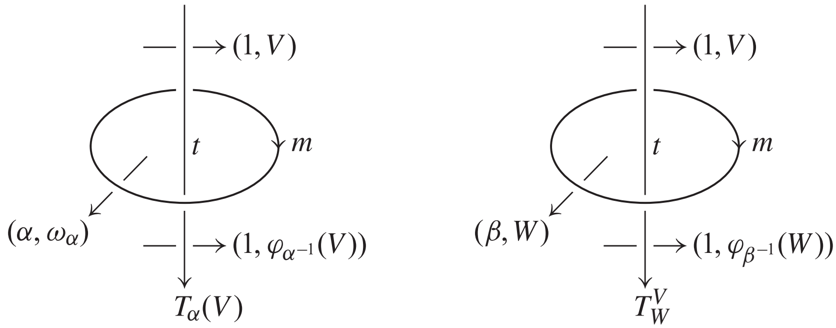


Figure VII.1. The tangles $T_\alpha(V)$ and T_W^V.

We claim that

$$F(T_W^V)\, F(T_\alpha(\varphi_{\beta^{-1}}(V))) = \dim(W)\, F(T_{\beta\alpha}(V)) \colon \varphi_{\alpha^{-1}\beta^{-1}}(V) \to V. \qquad (5.4.a)$$

We first prove this equality and then deduce the claim of the lemma.

Gluing T_W^V on the top of $T_\alpha(\varphi_{\beta^{-1}}(V))$ we obtain a colored G-tangle

$$\widetilde{T} = T_W^V \circ T_\alpha(\varphi_{\beta^{-1}}(V)).$$

Geometrically, $\widetilde{T}$ consists of a vertical interval with two meridians and all framings zero. Clearly,

$$F(\widetilde{T}) = F(T_W^V) \circ F(T_\alpha(\varphi_{\beta^{-1}}(V))).$$

Sliding the upper circle of $\widetilde{T}$ along the lower circle, we can transform $\widetilde{T}$ into a disjoint union of $T_{\beta\alpha}(V)$ with the colored G-knot (or rather unknot) ℓ_W^β in $\mathbb{R}^2 \times [0, 1]$ represented by a plane circle labeled with (β, W). (An explicit splitting of this handle sliding as a composition of Kirby–Fenn–Rourke moves is given in [Tu2], p. 93). As in the proof of Theorem 2.3,

$$F(\widetilde{T}) = F(\ell_W^\beta \amalg T_{\beta\alpha}(V)) = F(\ell_W^\beta)\, F(T_{\beta\alpha}(V)) = \dim(W)\, F(T_{\beta\alpha}(V)).$$

This implies (5.4.a).

Now we can prove the claim of the lemma. If $V = \mathbb{1}$, then formula (2.2.f) in Chapter VI implies that we can push the circle component of $T_\alpha(V)$ across the segment component keeping the operator invariant. Therefore $F(T_\alpha(V)) = d_\alpha \,\mathrm{id}_{\mathbb{1}}$ where $d_\alpha \in K$ is defined by (1.7.b). The equality $d_\alpha = \mathcal{D}^2$ proven in Section 1.7 implies that $F(T_\alpha(V)) = \mathcal{D}^2 \,\mathrm{id}_{\mathbb{1}}$.

It remains to show that if V is not isomorphic to $\mathbb{1}$, then $F(T_\alpha(V)) = 0$. Let I be the set of isomorphism classes of simple objects in $\mathcal{C}_1$ and let $\{V_i \in \mathcal{C}_1\}_{i\in I}$ be representatives of these classes. We can assume that $V = V_i$ for a certain $i \in I$. For $W = V_j$ with $j \in I$, we compute $F(T_W^V) \in \mathrm{End}(V)$ as follows. Since V is simple, $F(T_W^V) = k \,\mathrm{id}_V$ with $k \in K$. The closure of T_W^V is the Hopf link whose components are colored with $V = V_i$ and $W = V_j$. Therefore $k \dim(i) = S_{i,j}$ where $\dim(i) = \dim(V_i)$. Hence $k = (\dim(i))^{-1} S_{i,j}$ where we use the invertibility of $\dim(i)$ (Lemma 1.5). Substituting this in (5.4.a) (for $\beta = 1$) we obtain

$$(\dim(i))^{-1} S_{i,j}\, F(T_\alpha(V)) = \dim(j)\, F(T_\alpha(V)).$$

This holds for all $j \in I$ and implies that

$$\mathcal{D}^2 F(T_\alpha(V)) = \sum_{j\in I} (\dim(j))^2\, F(T_\alpha(V)) = (\dim(i))^{-1} \sum_{j\in I} \dim(j)\, S_{i,j}\, F(T_\alpha(V)).$$

Since $V = V_i$ is not isomorphic to $\mathbb{1}$, we have $\sum_{j\in I} \dim(j) S_{i,j} = 0$ (see [Tu2], formula (3.8.b)). Therefore $\mathcal{D}^2 F(T_\alpha(V)) = 0$. We know that $\mathcal{D}$ is invertible in K. Hence $F(T_\alpha(V)) = 0$. □

Chapter VIII
Miscellaneous algebra

VIII.1 Hopf G-coalgebras

1.1 Hopf algebras. We begin by recalling the standard definitions of quasitriangular and ribbon Hopf algebras; see, for instance, [KRT], [Tu2]. A *Hopf algebra* over K is a tuple $(A, \Delta, \varepsilon, s)$, where A is an associative K-algebra with unit 1_A, $\Delta\colon A \to A^{\otimes 2} = A \otimes A$ and $\varepsilon\colon A \to K$ are algebra homomorphisms, $S\colon A \to A$ is an algebra anti-automorphism such that Δ is coassociative and

$$(\mathrm{id}_A \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A\colon A \to A,$$
$$m(S \otimes \mathrm{id}_A)\Delta = m(\mathrm{id}_A \otimes S)\Delta = 1_A\, \varepsilon\colon A \to A,$$

where m is multiplication in A. Here and below by algebra (anti-)homomorphisms, we mean algebra (anti-)homomorphisms carrying the unit to the unit.

Let $A = (A, \Delta, \varepsilon, s)$ be a Hopf algebra over K. Denote the flip (permutation) $A^{\otimes 2} \to A^{\otimes 2}$ by σ. A pair $(A, R \in A^{\otimes 2})$ is a *quasitriangular Hopf algebra* if R is invertible in $A^{\otimes 2}$ and satisfies the following identities:

$$\sigma(\Delta(a)) = R\,\Delta(a)R^{-1} \quad \text{for all } a \in A,$$
$$(\mathrm{id}_A \otimes \Delta)(R) = R_{13}R_{12},$$
$$(\Delta \otimes \mathrm{id}_A)(R) = R_{13}R_{23},$$

where R_{12}, R_{13}, R_{23} are elements of $A^{\otimes 3}$ defined by

$$R_{12} = R \otimes 1_A, \quad R_{23} = 1_A \otimes R,$$
$$R_{13} = (\mathrm{id}_A \otimes \sigma)(R_{12}) = (\sigma \otimes \mathrm{id}_A)(R_{23}).$$

The identities above imply the Yang–Baxter equality

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Let (A, R) be a quasitriangular Hopf algebra. A triple $(A, R, \theta \in A)$ is a *ribbon Hopf algebra* if θ is an invertible element of the center of A such that $S(\theta) = \theta$ and $\Delta(\theta) = (\theta \otimes \theta)\,\sigma(R)R$. If $A = (A, R, \theta)$ is a ribbon Hopf algebra, then $(A, \sigma(R^{-1}), \theta^{-1})$ is also a ribbon Hopf algebra.

1.2 Hopf G-coalgebras. Let G be a group. The notion of a G-coalgebra is dual to the notion of a (unital) G-graded algebra. By a *G-coalgebra* over K, we mean a family of K-modules $\{A_\alpha\}_{\alpha\in G}$ endowed with a K-homomorphism $\varepsilon_1\colon A_1 \to K$ (the *counit*) and a family of K-homomorphisms (the *comultiplication*)

$$\Delta = \{\Delta_{\alpha,\beta}\colon A_{\alpha\beta} \to A_\alpha \otimes A_\beta\}_{\alpha,\beta\in G}$$

satisfying the following two axioms:

(1.2.1) Δ is coassociative in the sense that for any $\alpha, \beta, \gamma \in G$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{A_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{A_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}\colon A_{\alpha\beta\gamma} \to A_\alpha \otimes A_\beta \otimes A_\gamma,$$

(1.2.2) for any $\alpha \in G$,

$$(\mathrm{id}_{A_\alpha} \otimes \varepsilon_1)\Delta_{\alpha,1} = (\varepsilon_1 \otimes \mathrm{id}_{A_\alpha})\Delta_{1,\alpha} = \mathrm{id}_{A_\alpha}\colon A_\alpha \to A_\alpha.$$

A *Hopf G-coalgebra* over K is a G-coalgebra (A, Δ, ε), where each A_α is an associative K-algebra with multiplication m_α and unit 1_α, endowed with a family of algebra anti-isomorphisms $S = \{S_\alpha\colon A_\alpha \to A_{\alpha^{-1}}\}_{\alpha\in G}$ (the *antipode*) such that

(1.2.3) the comultiplication $\Delta_{\alpha,\beta}\colon A_{\alpha\beta} \to A_\alpha \otimes A_\beta$ is an algebra homomorphism for all $\alpha, \beta \in G$;

(1.2.4) the counit $\varepsilon_1\colon A_1 \to K$ is an algebra homomorphism;

(1.2.5) for any $\alpha \in G$,

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{A_\alpha})\Delta_{\alpha^{-1},\alpha} = m_\alpha(\mathrm{id}_{A_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}} = 1_\alpha\, \varepsilon_1\colon A_1 \to A_\alpha.$$

As a part of our conditions, for all $\alpha, \beta \in G$,

$$\Delta_{\alpha,\beta}(1_{\alpha\beta}) = 1_\alpha \otimes 1_\beta, \quad \varepsilon_1(1_1) = 1_K, \quad S_\alpha(1_\alpha) = 1_{\alpha^{-1}}.$$

Note also that the tuple $(A_1, \Delta_{1,1}, \varepsilon_1, S_1)$ is a Hopf algebra in the usual sense of the word. We call it the *neutral component* of the Hopf G-coalgebra A.

The notion of a Hopf G-coalgebra is not self-dual. It may be useful to study the dual notion of a Hopf G-algebra but we shall not need it. Hopf G-coalgebras may be viewed as multiplier Hopf algebras in the sense of Van Daele [VanD] with additional structure (the G-cograding). This allows one to deduce many properties of Hopf G-coalgebras from the work of Van Daele and his followers.

By a *crossed Hopf G-coalgebra* over K, we shall mean a Hopf G-coalgebra $(\{A_\alpha\}_{\alpha\in G}, \Delta, \varepsilon_1, S)$ endowed with a family of algebra isomorphisms

$$\varphi = \{\varphi_\alpha\colon A_\beta \to A_{\alpha\beta\alpha^{-1}}\}_{\alpha,\beta\in G}$$

such that

(1.2.6) each φ_α preserves the counit, the antipode, and the comultiplication, i.e., for any $\alpha, \beta, \gamma \in G$,

$$\varepsilon_1\, \varphi_\alpha|_{A_1} = \varepsilon_1,$$
$$\varphi_\alpha S_\beta = S_{\alpha\beta\alpha^{-1}}\varphi_\alpha\colon A_\beta \to A_{\alpha\beta^{-1}\alpha^{-1}},$$
$$(\varphi_\alpha \otimes \varphi_\alpha)\Delta_{\beta,\gamma} = \Delta_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}\, \varphi_\alpha\colon A_{\beta\gamma} \to A_{\alpha\beta\alpha^{-1}} \otimes A_{\alpha\gamma\alpha^{-1}};$$

(1.2.7) φ is an action of G, i.e., $\varphi_{\alpha\alpha'} = \varphi_\alpha \varphi_{\alpha'}$ for all $\alpha, \alpha' \in G$.

Note that $\varphi_\alpha(1_\beta) = 1_{\alpha\beta\alpha^{-1}}$ for all $\alpha, \beta \in G$ and $\varphi_\alpha(A_1) = A_1$. Restricting φ to A_1 we obtain an action of G on A_1 by Hopf algebra automorphisms.

We end this subsection with two examples of crossed Hopf G-coalgebras. Both examples are derived from an action of G on a Hopf algebra $(A, \Delta, \varepsilon, S)$ over K by Hopf algebra automorphisms. Set $A^G = \{A_\alpha\}_{\alpha\in G}$ where for each $\alpha \in G$, the algebra A_α is a copy of A. Fix an identification isomorphism of algebras $i_\alpha\colon A \to A_\alpha$. For $\alpha, \beta \in G$, we define $\Delta_{\alpha,\beta}\colon A_{\alpha\beta} \to A_\alpha \otimes A_\beta$ by

$$\Delta_{\alpha,\beta}(i_{\alpha\beta}(a)) = \sum_{(a)} i_\alpha(a') \otimes i_\beta(a''),$$

where $a \in A$ and $\Delta(a) = \sum_{(a)} a' \otimes a''$ is the given comultiplication in A written in Sweedler's sigma notation. The counit $\varepsilon_1\colon A_1 \to K$ is defined by $\varepsilon_1(i_1(a)) = \varepsilon(a) \in K$ for $a \in A$. For $\alpha \in G$, the antipode $S_\alpha\colon A_\alpha \to A_{\alpha^{-1}}$ is given by

$$S_\alpha(i_\alpha(a)) = i_{\alpha^{-1}}(S(a)),$$

where $a \in A$. For $\alpha, \beta \in G$, the homomorphism $\varphi_\alpha\colon A_\beta \to A_{\alpha\beta\alpha^{-1}}$ is defined by $\varphi_\alpha(i_\beta(a)) = i_{\alpha\beta\alpha^{-1}}(\alpha(a))$. The axioms of a crossed Hopf G-coalgebra for A^G follow directly from the definitions.

The second example differs only by the definitions of the comultiplication and the antipode. Let $\overline{A}^G$ be the same family of algebras $\{A_\alpha = A\}_{\alpha\in G}$ with the same counit and same action φ of G and with comultiplication $\overline{\Delta}_{\alpha,\beta}\colon A_{\alpha\beta} \to A_\alpha \otimes A_\beta$ and antipode $\overline{S}_\alpha\colon A_\alpha \to A_{\alpha^{-1}}$ such that for $a \in A$,

$$\overline{\Delta}_{\alpha,\beta}(i_{\alpha\beta}(a)) = \sum_{(a)} i_\alpha(\beta(a')) \otimes i_\beta(a''),$$

$$\overline{S}_\alpha(i_\alpha(a)) = i_{\alpha^{-1}}(\alpha(S(a))) = i_{\alpha^{-1}}(S(\alpha(a))).$$

The axioms of a crossed Hopf G-coalgebra for $\overline{A}^G$ follow from the definitions. Both A^G and $\overline{A}^G$ are extensions of A since $A^G_1 = \overline{A}^G_1 = A_1$ as Hopf algebras.

In particular, if G is a Lie group with Lie algebra $\mathfrak{g}$, then the universal enveloping algebra $U(\mathfrak{g})$ has a canonical structure of a Hopf algebra and G acts on $U(\mathfrak{g})$ by Hopf algebra automorphisms induced by the group conjugation. The constructions above give crossed Hopf G-algebras

$$(U(\mathfrak{g}))^G = \{U(\mathfrak{g})_\alpha\}_{\alpha\in G} \quad \text{and} \quad \overline{(U(\mathfrak{g}))}^G = \{U(\mathfrak{g})_\alpha\}_{\alpha\in G},$$

where each $U(\mathfrak{g})_\alpha$ is a copy of $U(\mathfrak{g})$ sitting at $\alpha \in G$.

1.3 Quasitriangular Hopf G-coalgebras. Let $A = (\{A_\alpha\}, \Delta, \varepsilon_1, S, \varphi)$ be a crossed Hopf G-coalgebra. A *universal R-matrix* in A is a family of invertible elements

$$R = \{R_{\alpha,\beta} \in A_\alpha \otimes A_\beta\}_{\alpha,\beta\in G} \tag{1.3.a}$$

satisfying the following conditions:

(1.3.1) for any $\alpha, \beta \in G$ and $a \in A_{\alpha\beta}$,

$$R_{\alpha,\beta}\, \Delta_{\alpha,\beta}(a) = \sigma_{\beta,\alpha}((\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{A_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(a))\, R_{\alpha,\beta},$$

where $\sigma_{\beta,\alpha}$ is the flip $A_\beta \otimes A_\alpha \to A_\alpha \otimes A_\beta$;

(1.3.2) for any $\alpha, \beta, \gamma \in G$,

$$(\mathrm{id}_{A_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3}\, (R_{\alpha,\beta})_{12\gamma},$$

and

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{A_\gamma})(R_{\alpha\beta,\gamma}) = ((\varphi_\beta \otimes \mathrm{id}_{A_\gamma})(R_{\beta^{-1}\alpha\beta,\gamma}))_{1\beta 3}\, (R_{\beta,\gamma})_{\alpha 23},$$

where for K-modules P, Q and $r = \sum_j p_j \otimes q_j \in P \otimes Q$ we set

$$r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes A_\gamma,$$
$$r_{\alpha 23} = 1_\alpha \otimes r \in A_\alpha \otimes P \otimes Q,$$
$$r_{1\beta 3} = \sum_j p_j \otimes 1_\beta \otimes q_j \in P \otimes A_\beta \otimes Q;$$

(1.3.3) the family (1.3.a) is invariant under the homomorphisms φ_α, i.e., for all α, β, $\gamma \in G$,

$$(\varphi_\alpha \otimes \varphi_\alpha)(R_{\beta,\gamma}) = R_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}.$$

A crossed Hopf G-coalgebra endowed with a universal R-matrix is said to be *quasitriangular*. It is easy to deduce from (1.3.1)–(1.3.3) the following Yang–Baxter equality for R:

$$(R_{\alpha,\beta})_{12\gamma}\, ((\varphi_\beta \otimes \mathrm{id}_{A_\gamma})(R_{\beta^{-1}\alpha\beta,\gamma}))_{1\beta 3}\, (R_{\beta,\gamma})_{\alpha 23} = (R_{\beta,\gamma})_{\alpha 23}\, (R_{\alpha,\gamma})_{1\beta 3}\, (R_{\alpha,\beta})_{12\gamma}.$$

1.4 Ribbon Hopf G-coalgebras. Let A be a quasitriangular crossed Hopf G-coalgebra with universal R-matrix (1.3.a). A *twist* in A is a collection of invertible elements $\{\theta_\alpha \in A_\alpha\}_{\alpha \in G}$ such that

(1.4.1) $\varphi_\alpha(a) = \theta_\alpha^{-1} a \theta_\alpha$ for all $\alpha \in G$ and $a \in A_\alpha$;

(1.4.2) $S_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}}$ for all $\alpha \in G$;

(1.4.3) for all $\alpha, \beta \in G$,

$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta}) = (\theta_\alpha \otimes \theta_\beta)\, \sigma_{\beta,\alpha}((\mathrm{id}_{A_\beta} \otimes \varphi_\alpha) R_{\beta,\alpha})\, R_{\alpha,\beta};$$

(1.4.4) $\varphi_\alpha(\theta_\beta) = \theta_{\alpha\beta\alpha^{-1}}$ for all $\alpha, \beta \in G$.

A quasitriangular crossed Hopf G-coalgebra endowed with a twist is said to be *ribbon*.

It follows from the definitions that the neutral component A_1 of A endowed with $R_{1,1} \in A_1 \otimes A_1$ and $\theta_1 \in A_1$ is a ribbon Hopf algebra in the sense of Section 1.1. In particular, the equality $\varphi_1 = \mathrm{id}$ implies that θ_1 lies in the center of A_1.

For $G = 1$, the notions introduced in Sections 1.3 and 1.4 boil down to the standard notions of quasitriangular and ribbon Hopf algebras.

1.5 Examples. We give two examples of a ribbon crossed Hopf G-coalgebra. Let $(A, \Delta, \varepsilon, S, R, \theta)$ be a ribbon Hopf algebra over K. An element $\alpha \in A$ is *group-like* if $\Delta(\alpha) = \alpha \otimes \alpha$ and $\varepsilon(\alpha) = 1 \in K$. Any group-like α is invertible and $S(\alpha) = \alpha^{-1}$. The group-like elements of A form a group $G = G(A)$ under multiplication in A. For $\alpha \in G$, the formula $a \mapsto \alpha a \alpha^{-1}$ with $a \in A$ defines a Hopf algebra automorphism of A. This gives an action of G on A by Hopf algebra automorphisms. Applying the constructions of Section 1.2 to this action we obtain crossed Hopf G-coalgebras A^G and $\overline{A}^G$. We define a universal R-matrix and twist in A^G by

$$R_{\alpha,\beta} = (i_\alpha \otimes i_\beta)((1_A \otimes \alpha^{-1})\, R) \in A_\alpha \otimes A_\beta \quad \text{and} \quad \theta_\alpha = i_\alpha(\theta\alpha^{-1}) \in A_\alpha,$$

where $\alpha, \beta \in G$. We define a universal R-matrix and twist in $\overline{A}^G$ by

$$\overline{R}_{\alpha,\beta} = (i_\alpha \otimes i_\beta)(R\,(\beta^{-1} \otimes 1_A)) \in A_\alpha \otimes A_\beta \quad \text{and} \quad \bar{\theta}_\alpha = i_\alpha(\theta\alpha^{-1}) \in A_\alpha,$$

where $\alpha, \beta \in G$. A direct computation shows that $(A^G, \{R_{\alpha,\beta}\}_{\alpha,\beta\in G}, \{\theta_\alpha\}_{\alpha\in G})$ and $(\overline{A}^G, \{\overline{R}_{\alpha,\beta}\}_{\alpha,\beta\in G}, \{\bar{\theta}_\alpha\}_{\alpha\in G})$ are ribbon crossed Hopf G-coalgebras.

Group-like elements of a quantum universal enveloping algebra $A = U_q(\mathfrak{g})$ are well known. For instance, if $\mathfrak{g} = sl(N+1)$ and q is generic, then $G(A) = \mathbb{Z}^N$ is a free abelian group of rank N generated by the canonical group-like elements $K_1, \dots, K_N \in A$. If q is a primitive root of unity of order ℓ, then one usually considers a version A^{res} of $A = U_q((N+1))$ with $K_i^\ell = 1$ for all $i = 1, \dots, N$ (see [KRT]). Then $G(A^{\text{res}}) = (\mathbb{Z}/\ell\mathbb{Z})^N$.

1.6 Operations on Hopf group-coalgebras. Given a group homomorphism $q\colon G' \to G$, we can pull back a Hopf G-coalgebra A along q. This gives a Hopf G'-algebra $A' = q^*(A)$ defined by $A'_\alpha = A_{q(\alpha)}$ for any $\alpha \in G'$. If A is crossed (resp. quasitriangular, ribbon), then A' has the structure of a crossed (resp. quasitriangular, ribbon) Hopf G'-algebra obtained by lifting the data from A to A' in the obvious way. For example, taking $G = 1$ and choosing as A any ribbon Hopf algebra, we obtain a ribbon Hopf G'-algebra for any group G'.

Given a crossed Hopf G-coalgebra $A = (\{A_\alpha\}_{\alpha\in G}, \Delta, \varepsilon_1, S, \varphi)$, we define its mirror $\overline{A} = (\{\overline{A}_\alpha\}_{\alpha\in G}, \overline{\Delta}, \bar{\varepsilon}_1, \overline{S}, \bar{\varphi})$. For $\alpha \in G$, set $\overline{A}_\alpha = A_{\alpha^{-1}}$. For $a \in \overline{A}_{\alpha\beta}$, set

$$\begin{aligned}\overline{\Delta}_{\alpha,\beta}(a) &= (\varphi_\beta \otimes \mathrm{id}_{A_{\beta^{-1}}})\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}(a)\\ &= (\mathrm{id}_{A_{\alpha^{-1}}} \otimes \varphi_{\beta^{-1}})\Delta_{\alpha^{-1},\beta^{-1}}\varphi_\beta(a) \in A_{\alpha^{-1}} \otimes A_{\beta^{-1}} = \overline{A}_\alpha \otimes \overline{A}_\beta.\end{aligned}$$

This defines a comultiplication $\overline{\Delta}_{\alpha,\beta}\colon \overline{A}_{\alpha\beta} \to \overline{A}_\alpha \otimes \overline{A}_\beta$. Set

$$\bar{\varepsilon}_1 = \varepsilon_1\colon \overline{A}_1 = A_1 \to K.$$

For $\alpha \in G$, set

$$\overline{S}_\alpha = \varphi_\alpha S_{\alpha^{-1}}\colon \overline{A}_\alpha = A_{\alpha^{-1}} \to A_\alpha = \overline{A}_{\alpha^{-1}}.$$

Finally, set $\bar{\varphi}_\alpha = \varphi_\alpha$ for all $\alpha \in G$. By a direct computation it can be shown that $\overline{A} = (\{\overline{A}_\alpha\}_{\alpha\in G}, \overline{\Delta}, \bar{\varepsilon}_1, \overline{S}, \bar{\varphi})$ is a crossed Hopf G-coalgebra. Moreover, if R, θ are a universal R-matrix and a twist in A, respectively, then the formulas

$$\overline{R}_{\alpha,\beta} = (\sigma_{\beta^{-1},\alpha^{-1}}(R_{\beta^{-1},\alpha^{-1}}))^{-1} \in \overline{A}_\alpha \otimes \overline{A}_\beta, \quad \bar{\theta}_\alpha = (\theta_{\alpha^{-1}})^{-1} \in \overline{A}_\alpha$$

define a universal R-matrix and a twist in $\overline{A}$. It is easy to see that $\overline{\overline{A}} = A$.

For example, the crossed Hopf G-coalgebras A^G and $\overline{A}^G$ defined in Section 1.2 are mirrors of each other. The ribbon Hopf G-coalgebras A^G and $\tilde{A}^G$ constructed in Section 1.5 are related as follows: $\overline{A}^G$ is the mirror of B^G, where $B = (A, (\sigma(R))^{-1}, \theta^{-1} \in A)$ and $G = G(A) = G(B)$.

1.7 Category of representations. It is well known that the category of representations of a quasitriangular (resp. ribbon) Hopf algebra is a braided (resp. ribbon) monoidal tensor category. Following this line, we associate with every Hopf G-coalgebra $A = (\{A_\alpha\}, \Delta, \varepsilon_1, S)$ a category of representations $\mathrm{Rep}(A)$ with a natural structure of a G-category. If A is crossed (resp. quasitriangular, ribbon) then $\mathrm{Rep}(A)$ is crossed (resp. braided, ribbon).

By an A_α*-module*, we mean a left A_α-module whose underlying K-module is projective of finite type. (The unit of A_α is supposed to act as the identity.) The category $\mathrm{Rep}(A)$ is the disjoint union of the categories $\{\mathrm{Rep}(A_\alpha)\}_{\alpha\in G}$, where $\mathrm{Rep}(A_\alpha)$ is the category of A_α-modules and A_α-homomorphisms. The tensor product and the unit object $\mathbb{1} = K$ in $\mathrm{Rep}(A)$ are defined in the usual way using the co-multiplication Δ and the co-unit ε_1. The associativity morphisms are the standard identification isomorphisms of modules $(U \otimes V) \otimes W = U \otimes (V \otimes W)$; they will be suppressed from the notation. The same applies to the structural morphisms l, r which are the standard identifications $U \otimes K = U = K \otimes U$.

For $U \in \mathrm{Rep}(A_\alpha)$, we set

$$U^* = \mathrm{Hom}_K(U, K) \in \mathrm{Rep}(A_{\alpha^{-1}}),$$

where each $a \in A_{\alpha^{-1}}$ acts as the transpose of $x \mapsto S_{\alpha^{-1}}(a)(x) \colon U \to U$. The duality morphism $d_U \colon U^* \otimes U \to \mathbb{1} = K$ is the evaluation pairing; it determines b_U in the usual way, cf. [Tu2], Chapter XI. The equalities (1.2.2) imply that d_U, b_U are A_1-linear.

The automorphism φ_α of A defines an automorphism Φ_α of $\mathrm{Rep}(A)$. If $U \in \mathrm{Rep}(A_\beta)$, then $\Phi_\alpha(U)$ has the same underlying K-module as U and each $a \in A_{\alpha\beta\alpha^{-1}}$ acts as multiplication by $\varphi_\alpha^{-1}(a) \in A_\beta$. An A_β-homomorphism $U \to U'$ is mapped to itself viewed as an $A_{\alpha\beta\alpha^{-1}}$-homomorphism. It is easy to check that $\mathrm{Rep}(A)$ is a crossed G-category.

A universal R-matrix (1.3.a) in A induces a braiding in $\mathrm{Rep}(A)$ as follows. For $U \in \mathrm{Rep}(A_\alpha)$ and $V \in \mathrm{Rep}(A_\beta)$, the braiding $c_{V,W} \colon V \otimes W \to {}^V W \otimes V$ is the composition of multiplication by $R_{\alpha,\beta}$, permutation $V \otimes W \to W \otimes V$ and the K-isomorphism $W \otimes V = {}^V W \otimes V$ which comes from the fact that $W = {}^V W$ as K-modules. The axioms of a universal R-matrix imply that $\{c_{V,W}\}_{V,W}$ is a braiding.

A twist $\{\theta_\alpha\}$ in A induces a twist in $\mathrm{Rep}(A)$: for any A_α-module V, the morphism $\theta_V : V \to {}^V V$ is the composition of multiplication by $\theta_\alpha \in A_\alpha$ and the identity map $V = {}^V V$. Conditions (1.4.1) and (1.4.4) imply that θ_V is A_α-linear. Condition VI.2.3.1 follows from the definitions, Conditions VI.2.3.2–VI.2.3.4 follow from (1.4.2)–(1.4.4), respectively. Thus, $\mathrm{Rep}(A)$ is a ribbon crossed G-category.

As an exercise, the reader may check that $\overline{\mathrm{Rep}(A)} = \mathrm{Rep}(\overline{A})$.

1.8 Remarks. 1. The idea of a Hopf group-coalgebra comes from the following observation. Consider a topological group G. For $\alpha \in G$, denote by $C_\alpha = C_\alpha(G)$ the algebra of germs of continuous functions at $\alpha \in G$. The group multiplication $G \times G \to G$ induces an algebra homomorphism $C_1 \to C_1 \hat{\otimes} C_1$ which turns C_1 into a topological Hopf algebra. Similarly, the group multiplication in G induces an algebra homomorphism $\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \hat{\otimes} C_\beta$ for any $\alpha, \beta \in G$. This turns the system of algebras $\{C_\alpha\}_{\alpha \in G}$ into a (topological) Hopf G-coalgebra.

In this example we can compute $\Delta_{\alpha,\beta}$ via $\Delta_{1,1}$ and the adjoint action of G on C_1 as follows. Observe that left multiplication by α induces an algebra isomorphism, $i_\alpha : C_1 \to C_\alpha$. Let x, y be two elements of G close to $1 \in G$. Then $\alpha x, \beta y$ are close to α, β, respectively. For $f \in C_1$,

$$\begin{aligned}(i_\alpha^{-1} \otimes i_\beta^{-1})\Delta_{\alpha,\beta}(i_{\alpha\beta}(f))(x, y) &= \Delta_{\alpha,\beta}(i_{\alpha\beta}(f))(\alpha x, \beta y) \\ &= i_{\alpha\beta}(f)(\alpha x \beta y) \\ &= f((\alpha\beta)^{-1}\alpha x \beta y) \\ &= f(\beta^{-1} x \beta y) \\ &= \Delta_{1,1}(f)(\beta^{-1} x \beta, y).\end{aligned}$$

This computation suggests the second example of Hopf G-coalgebras in Section 1.2.

2. In analogy with [RT] one can define a notion of a modular crossed Hopf G-coalgebra. The category of representations of such a coalgebra is a modular crossed G-category and thus gives a 3-dimensional HQFT.

3. There is a number of constructions of topological invariants of 3-manifolds from Hopf algebras; see [TV], [BW2], [He], [Ku]. All these constructions can be generalized to 3-dimensional G-manifolds, see Appendices 2 and 7.

4. The transfer defined for G-algebras in Section II.4 and for group-categories in Section 3 below can also be defined for crossed, quasitriangular, and ribbon Hopf group-coalgebras. We leave the details to the reader.

5. There is a special case of the definitions above where all the crossed isomorphisms $\{\varphi_\alpha\}_{\alpha \in G}$ are the identity maps. Namely, assume that A is a Hopf G-coalgebra over an abelian group G. Then the trivial homomorphism $\varphi = 1 : G \to \mathrm{Aut}(A)$ turns A into a crossed Hopf G-coalgebra. In this case the definitions of the universal R-matrix and the twist in A simplify considerably. Such quasitriangular crossed Hopf G-coalgebras were first considered by T. Ohtsuki [Oh1], [Oh2]. He calls them *colored Hopf algebras* and derives examples of such algebras from $U_q(sl_2)$.

VIII.2 Canonical extensions

We show that any monoidal category canonically extends to a crossed group-category in the sense of Section VI.2. Similarly, a braided (resp. ribbon) monoidal category canonically extends to a braided (resp. ribbon) crossed group-category.

Throughout this section, the symbol $\mathcal{C}$ denotes a K-additive monoidal category with left duality.

2.1 Crossed extensions of $\mathcal{C}$. Observe that $\mathcal{C}$ is a $\{1\}$-category, where $\{1\}$ is the trivial group. Given a group G, the category $\mathcal{C}$ gives rise to a G-category $\mathcal{C}^G$ obtained by pulling back $\mathcal{C}$ along the trivial homomorphism $G \to \{1\}$. Thus, $\mathcal{C}^G = \coprod_{\alpha \in G} \mathcal{C}^G_\alpha$, where the objects of $\mathcal{C}^G_\alpha$ are arbitrary pairs $(U \in \mathcal{C}, \alpha \in G)$. A morphism of such pairs $(U, \alpha) \to (V, \beta)$ is $0 \in \mathrm{Hom}_\mathcal{C}(U, V)$ if $\alpha \neq \beta$ and any element of $\mathrm{Hom}_\mathcal{C}(U, V)$ if $\alpha = \beta$. The operations on objects and the unit object are defined by

$$(U, \alpha) \otimes (V, \beta) = (U \otimes V, \alpha\beta), \quad (U, \alpha)^* = (U^*, \alpha^{-1}), \quad \mathbb{1}_{\mathcal{C}^G} = (\mathbb{1}_\mathcal{C}, 1).$$

The structural morphisms (1.1.a), (1.1.b), (1.1.e) of Chapter VI, as well as composition and tensor product of morphisms are induced by the corresponding data in $\mathcal{C}$ in the obvious way.

Assume that G acts on $\mathcal{C}$ by automorphisms, i.e., we have a homomorphism from G to the group $\mathrm{Aut}(\mathcal{C})$ of automorphisms of $\mathcal{C}$ defined in Section VI.2.1 Then $\mathcal{C}^G$ acquires a structure of a crossed G-category as follows. For $\alpha \in G$ and $(V, \beta) \in \mathcal{C}^G_\beta$, set

$$\varphi_\alpha(V, \beta) = (\alpha(V), \alpha\beta\alpha^{-1}) \in \mathcal{C}^G_{\alpha\beta\alpha^{-1}}.$$

For a morphism $f : (V, \beta) \to (W, \gamma)$ in $\mathcal{C}^G$ with $f \in \mathrm{Hom}_\mathcal{C}(V, W)$, set

$$\varphi_\alpha(f) = \alpha(f) \in \mathrm{Hom}_\mathcal{C}(\alpha(V), \alpha(W)).$$

The axioms of a crossed G-category follow directly from the definitions.

Applying this construction to $G = \mathrm{Aut}(\mathcal{C})$, we obtain an extension of $\mathcal{C}$ to a crossed $\mathrm{Aut}(\mathcal{C})$-category.

2.2 The group $\mathrm{Aut}_0(\mathcal{C})$. Denote by $\mathrm{id}_\mathcal{C}$ the identity functor $\mathcal{C} \to \mathcal{C}$ carrying each object and each morphism of $\mathcal{C}$ into itself. Consider the group $G_0 = \mathrm{Aut}_0(\mathcal{C})$ formed by monoidal equivalences of $\mathrm{id}_\mathcal{C}$ to automorphisms of $\mathcal{C}$. An element of G_0 is a pair ($\alpha \in \mathrm{Aut}(\mathcal{C})$, an invertible monoidal morphism of functors $F : \mathrm{id}_\mathcal{C} \to \alpha$). The latter means that for each object $U \in \mathcal{C}$ we have an invertible morphism $F_U : U \to \alpha(U)$ in $\mathcal{C}$ such that

(i) for any morphism $f: U \to V$ in $\mathcal{C}$ the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle F_U}\downarrow & & \downarrow{\scriptstyle F_V} \\ \alpha(U) & \xrightarrow{\alpha(f)} & \alpha(V); \end{array}$$

(ii) $F_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ and $F_{U \otimes V} = F_U \otimes F_V$ for any $U, V \in \mathcal{C}$.

The product of $(\alpha, F), (\alpha', F') \in G_0$ is the pair $(\alpha\alpha', FF')$, where for $U \in \mathcal{C}$, we set $(FF')_U = F_{\alpha'(U)} F'_U : U \to (\alpha\alpha')(U)$. The set G_0 with this multiplication is a group with neutral element $(\alpha = \mathrm{id}_{\mathcal{C}}, F = \{\mathrm{id}_U\}_{U \in \mathcal{C}})$.

Forgetting F, we obtain a homomorphism $i: G_0 \to \mathrm{Aut}(\mathcal{C})$ whose image consists of automorphisms of $\mathcal{C}$ monoidally equivalent to $\mathrm{id}_{\mathcal{C}}$. The key property of this image is given by the following lemma.

2.2.1 Lemma. *If $\mathcal{C}$ is braided, then all elements of $i(G_0) \subset \mathrm{Aut}(\mathcal{C})$ preserve the braiding. If $\mathcal{C}$ is ribbon, then all elements of $i(G_0)$ preserve the twist.*

Proof. Let $\{c_{U,V}: U \otimes V \to V \otimes U\}_{U,V \in \mathcal{C}}$ be a braiding in $\mathcal{C}$ and $(\alpha, F) \in G_0$. For any $U, V \in \mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccc} U \otimes V & \xrightarrow{c_{U,V}} & V \otimes U \\ {\scriptstyle F_{U\otimes V}}\downarrow & & \downarrow{\scriptstyle F_{V\otimes U}} \\ \alpha(U) \otimes \alpha(V) & \xrightarrow{\alpha(c_{U,V})} & \alpha(V) \otimes \alpha(U). \end{array} \qquad (2.2.a)$$

By the naturality of the braiding,

$$F_{V \otimes U}\, c_{U,V} = (F_V \otimes F_U)\, c_{U,V} = c_{\alpha(U),\alpha(V)}(F_U \otimes F_V) = c_{\alpha(U),\alpha(V)} F_{U \otimes V}.$$

Thus, replacing in (2.2.a) the bottom arrow by $c_{\alpha(U),\alpha(V)}$, we obtain a commutative diagram. Since all arrows in (2.2.a) are invertible morphisms, $c_{\alpha(U),\alpha(V)} = \alpha(c_{U,V})$. Similarly, if $\{\theta_U: U \to U\}_{U \in \mathcal{C}}$ is a twist in $\mathcal{C}$, then for any $U \in \mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\theta_U} & U \\ {\scriptstyle F_U}\downarrow & & \downarrow{\scriptstyle F_U} \\ \alpha(U) & \xrightarrow{\alpha(\theta_U)} & \alpha(U). \end{array}$$

By the naturality of the twist, $F_U \theta_U = \theta_{\alpha(U)} F_U$. Hence $\alpha(\theta_U) = \theta_{\alpha(U)}$. □

2.3 Canonical extension of a braided category. Suppose additionally that the category $\mathcal{C}$ is braided and set $G_0 = \mathrm{Aut}_0(\mathcal{C})$. We define a canonical extension of

$\mathcal{C}$ to a braided crossed G_0-category. Applying the constructions of Section 2.1 to the homomorphism $i\colon G_0 \to \mathrm{Aut}(\mathcal{C})$ we obtain a crossed G-category $\widehat{\mathcal{C}} = \mathcal{C}^{G_0}$. The braiding $\{c_{U,V}\colon U \otimes V \to V \otimes U\}_{U,V \in \mathcal{C}}$ in $\mathcal{C}$ induces a braiding in $\widehat{\mathcal{C}}$ as follows. Let $u = (U,(\alpha,F)), v = (V,(\beta,H))$ be objects of $\widehat{\mathcal{C}}$, where $U, V \in \mathcal{C}$ and $(\alpha,F),(\beta,H) \in G$. Observe that

$$\begin{aligned} u \otimes v &= (U \otimes V, (\alpha,F)(\beta,H)),\\ \varphi_{(\alpha,F)}(v) &= (\alpha(V), (\alpha,F)(\beta,H)(\alpha,F)^{-1}),\\ {}^{u}v \otimes u &= \varphi_{(\alpha,F)}(v) \otimes u = (\alpha(V) \otimes U, (\alpha,F)(\beta,H)). \end{aligned}$$

The invertible morphism $(F_V \otimes \mathrm{id}_U)\, c_{U,V}\colon U \otimes V \to \alpha(V) \otimes U$ in $\mathcal{C}$ defines an invertible morphism $c_{u,v}\colon u \otimes v \to {}^{u}v \otimes u$ in $\widehat{\mathcal{C}}$.

2.3.1 Theorem. *The morphisms $\{c_{u,v}\colon u \otimes v \to {}^{u}v \otimes u \mid u, v \in \widehat{\mathcal{C}}\}$ form a braiding in the crossed G-category $\widehat{\mathcal{C}}$.*

If $\mathcal{C}$ is a ribbon category, then the twist $\{\theta_U\colon U \to U\}_{U \in \mathcal{C}}$ in $\mathcal{C}$ induces a twist in $\widehat{\mathcal{C}}$ as follows. Let $u = (U,(\alpha,F)) \in \widehat{\mathcal{C}}$. Then the invertible morphism $F_U \theta_U\colon U \to \alpha(U)$ in $\mathcal{C}$ defines an invertible morphism in $\widehat{\mathcal{C}}$

$$\theta_u\colon u \to {}^{u}u = \varphi_{(\alpha,F)}(u) = (\alpha(U),(\alpha,F)).$$

2.3.2 Theorem. *The morphisms $\{\theta_u \mid u \in \widehat{\mathcal{C}}\}$ form a twist in $\widehat{\mathcal{C}}$.*

The proof of Theorems 2.3.1 and 2.3.2 consists in a routine verification of the axioms, we leave it to the reader. Note that the neutral component of $\widehat{\mathcal{C}}$ is $\mathcal{C}$ with its original braiding and twist. Therefore if $\mathcal{C}$ is modular, then so is $\widehat{\mathcal{C}}$.

2.4 Remarks and examples. 1. Condition 2.2 (i) on a pair $(\alpha,F) \in \mathrm{Aut}_0(\mathcal{C})$ shows that the action of α on morphisms is completely determined by F. Setting $F(U) = \alpha(U)$ for $U \in \mathcal{C}$, we can reformulate the definition of $\mathrm{Aut}_0(\mathcal{C})$ entirely in terms of F. An element of $\mathrm{Aut}_0(\mathcal{C})$ is thus described as a pair (a bijection F from the set of objects of $\mathcal{C}$ into itself, a system of invertible morphisms $\{F_U\colon U \to F(U)\}_{U \in \mathcal{C}}$) such that

(i) $F(\mathbb{1}) = \mathbb{1}$ and $F_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$;

(ii) $F(U \otimes V) = F(U) \otimes F(V)$ and $F_{U \otimes V} = F_U \otimes F_V$, for any $U, V \in \mathcal{C}$;

(iii) for any $U \in \mathcal{C}$, we have $F(U^*) = (F(U))^*$ and $F_{U^*} = ((F_U)^*)^{-1}$, where $(F_U)^*\colon (F(U))^* \to U^*$ is the transpose of F_U.

2. Consider the ribbon crossed G-category $\mathcal{C} = \mathcal{C}(a,b,c,\theta)$ from Example VI.2.6, and assume additionally that the group G is abelian. Then $\varphi_\alpha = \mathrm{id}$ for all $\alpha \in G$, and so $\mathcal{C}$ is a ribbon monoidal category in the usual sense of the word. Any automorphism of $\mathcal{C}$ monoidally equivalent to $\mathrm{id}_{\mathcal{C}}$ is equal to $\mathrm{id}_{\mathcal{C}}$ since all non-zero morphisms in $\mathcal{C}$ are proportional to the identity morphisms of objects. An element $(\alpha,F) \in \mathrm{Aut}_0(\mathcal{C})$ is therefore completely determined by the map $U \mapsto F_U \in \mathrm{Hom}_{\mathcal{C}}(U,U) = K$, where

U runs over the elements of G. The inclusion $(\alpha, F) \in \mathrm{Aut}_0(\mathcal{C})$ is equivalent to the condition that this map is a group homomorphism from G to the group K^* of invertible elements of K. Hence, $\mathrm{Aut}_0(\mathcal{C}) = \mathrm{Hom}(G, K^*) = G^*$. By Section 2.3, $\mathcal{C}$ gives rise to a ribbon crossed G^*-category $\widehat{\mathcal{C}} = \mathcal{C}^{G^*}$.

3. Let G be a topological group and $\mathcal{C} = \mathrm{Rep}(G)$ be the category of G-modules (i.e., the category of linear representations of G in projective K-modules of finite rank and G-linear homomorphisms). It is clear that $\mathcal{C}$ is a K-additive monoidal category with left duality. This category is braided (in fact symmetric) with braiding given by the flips (permutations) $U \otimes V \to V \otimes U$. The category $\mathcal{C}$ is ribbon with $\theta_U = \mathrm{id}_U : U \to U$ for all $U \in \mathcal{C}$. We define a homomorphism

$$G \to \mathrm{Aut}_0(\mathcal{C}), \quad g \mapsto (\alpha^g, F^g), \tag{2.4.a}$$

as follows. For $g \in G$, the functor $\alpha^g : \mathcal{C} \to \mathcal{C}$ carries an arbitrary G-module $(U, \rho_U : G \to \mathrm{End}(U))$ into the same K-module U, where each $h \in G$ acts as $\rho_U(g^{-1}hg)$. The functor α^g carries each G-linear homomorphism to itself. It is clear that $\alpha^g \in \mathrm{Aut}(\mathcal{C})$. The morphism $F^g_U : U \to \alpha^g(U)$ in $\mathcal{C}$ carries any $u \in U$ to $\rho_U(g^{-1})(u) \in U$. The family $\{F^g_U\}_{U \in \mathcal{C}}$ satisfies the conditions of Section 2.2. Thus, $(\alpha^g, F^g) \in \mathrm{Aut}_0(\mathcal{C})$. Pulling back the canonical extension $\widehat{\mathcal{C}}$ of $\mathcal{C}$ along the homomorphism (2.4.a) we obtain a ribbon crossed G-category $(\mathrm{Rep}(G))^G$.

If G is a semisimple complex connected Lie group with Lie algebra $\mathfrak{g}$, then $\mathrm{Rep}(G) = \mathrm{Rep}(U(\mathfrak{g}))$. It would be interesting to find a quantum deformation of the ribbon crossed G-category $(\mathrm{Rep}(G))^G = (\mathrm{Rep}(U(\mathfrak{g})))^G$ generalizing the deformation $\mathrm{Rep}(U_q(\mathfrak{g}))$ of $\mathrm{Rep}(U(\mathfrak{g}))$ arising in the theory of quantum groups.

VIII.3 Transfer of categories

In this section, H is a subgroup of a group G. We derive from each H-category $\mathcal{C}$ a G-category $\widehat{\mathcal{C}}$ called the *transfer* of $\mathcal{C}$. If $\mathcal{C}$ is crossed (resp. braided, ribbon), then $\widehat{\mathcal{C}}$ is crossed (resp. braided, ribbon).

3.1 Transfer of H-categories. Consider an H-category $\mathcal{C} = \coprod_{h \in H} \mathcal{C}_h$. We construct its transfer $\widehat{\mathcal{C}}$ as follows. Fix a representative $\omega_i \in G$ for each right coset $i \in H \backslash G$ so that $i = H\omega_i \subset G$. For $\alpha \in G$, set

$$N(\alpha) = \{i \in H\backslash G \mid \omega_i \alpha \omega_i^{-1} \in H\}.$$

An object of $\widehat{\mathcal{C}}$ is a triple

$$(\alpha \in G, \text{ a subset (possibly empty) } A \text{ of } N(\alpha), \text{ a family } \{U_i \in \mathcal{C}_{\omega_i \alpha \omega_i^{-1}}\}_{i \in A}).$$

For such a triple $U = (\alpha, A, \{U_i\})$, set $\underline{U} = \alpha$ and $|U| = A$. The set of morphisms $U \to U'$ in $\widehat{\mathcal{C}}$ is defined by

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(U, U') = \begin{cases} 0 & \text{if } \underline{U} \neq \underline{U}', \\ \prod_{i \in |U| \cap |U'|} \mathrm{Hom}_{\mathcal{C}}(U_i, U'_i) & \text{if } \underline{U} = \underline{U}'. \end{cases}$$

Thus, in the case $\underline{U} = \underline{U}'$, a morphism $f : U \to U'$ in $\widehat{\mathcal{C}}$ is a family

$$\{f_i : U_i \to U'_i\}_{i \in |U| \cap |U'|}$$

of morphisms in $\mathcal{C}$. We view f_i as the i-th coordinate of f. If $\underline{U} \neq \underline{U}'$, then $f_i = 0$ for all $i \in |U| \cap |U'|$. The K-linear structure in $\mathrm{Hom}_{\widehat{\mathcal{C}}}(U, U')$ is coordinate-wise. The composition of two morphisms $f : U \to U'$ and $f' : U' \to U''$ in $\widehat{\mathcal{C}}$ is defined in the coordinates by

$$(f'f)_i = \begin{cases} f'_i f_i : U_i \to U''_i & \text{if } i \in |U| \cap |U'| \cap |U''|, \\ 0 : U_i \to U''_i & \text{if } i \in (|U| \cap |U''|) - |U'|. \end{cases}$$

The composition is associative because the composition of morphisms in $\mathcal{C}$ is associative and the composition of a zero morphism in $\mathcal{C}$ with any morphism is equal to zero. This defines $\widehat{\mathcal{C}}$ as a K-additive category. Clearly, $\widehat{\mathcal{C}} = \coprod_{\alpha \in G} \widehat{\mathcal{C}}_\alpha$, where $\widehat{\mathcal{C}}_\alpha$ is the full subcategory of $\widehat{\mathcal{C}}$ formed by the objects U with $\underline{U} = \alpha$.

The unit object of $\widehat{\mathcal{C}}$ is the triple $(1 \in G, A = H \backslash G, \{U_i = \mathbb{1}_{\mathcal{C}}\}_{i \in A})$. The duality and tensor product for objects of $\widehat{\mathcal{C}}$ are defined by

$$(\alpha, A, \{U_i\}_{i \in A})^* = (\alpha^{-1}, A, \{U_i^*\}_{i \in A}),$$
$$(\alpha, A, \{U_i\}_{i \in A}) \otimes (\beta, B, \{V_j\}_{j \in B}) = (\alpha\beta, A \cap B, \{U_i \otimes V_i\}_{i \in A \cap B}).$$

Note that $N(\alpha) = N(\alpha^{-1})$ and the inclusions $A \subset N(\alpha)$, $B \subset N(\beta)$ imply that $A \cap B \subset N(\alpha\beta)$. Clearly, $|U^*| = |U|$ and $|U \otimes V| = |U| \cap |V|$.

The tensor product of morphisms $f : U \to U'$ and $g : V \to V'$ is defined by

$$(f \otimes g)_i = f_i \otimes g_i : U_i \otimes V_i \to U'_i \otimes V'_i$$

for all $i \in |U| \cap |V| \cap |U'| \cap |V'|$. It is a simple exercise to check the identity $(f' \otimes g')(f \otimes g) = f'f \otimes g'g$.

The structural morphisms a, l, r, b, d in $\widehat{\mathcal{C}}$ are defined coordinate-wise and their coordinates are the corresponding structural morphisms in $\mathcal{C}$. In particular, for every $U \in \widehat{\mathcal{C}}$, we define $b_U : \mathbb{1} \to U \otimes U^*$ by $(b_U)_i = b_{U_i} : \mathbb{1} \to U_i \otimes (U_i)^*$ for all $i \in |U|$. Similarly, $(d_U)_i = d_{U_i}$, $(l_U)_i = l_{U_i}$, and $(r_U)_i = r_{U_i}$ for all $i \in |U|$. The associativity morphisms are defined by $(a_{U,V,W})_i = a_{U_i,V_i,W_i}$ for all indices $i \in |U| \cap |V| \cap |W|$. The naturality of a, l, r and the identities (1.1.c), (1.1.d), (1.1.f), (1.1.g) in Chapter VI follow from the corresponding properties of $\mathcal{C}$.

3.2 Transfer of crossed *H*-categories. The transfer $\widehat{\mathcal{C}}$ of a crossed H-category $(\mathcal{C}, \varphi\colon H \to \mathrm{Aut}(\mathcal{C}))$ is a crossed G-category as follows. We only need to define the action of G on $\widehat{\mathcal{C}}$. Fix the representatives $\omega_i \in i$ for $i \in H\backslash G$ used in the construction of $\widehat{\mathcal{C}}$. Consider the left action of G on $H\backslash G$ defined by $\alpha(i) = i\alpha^{-1}$ for $\alpha \in G$ and $i \in H\backslash G$. We have $H\omega_{\alpha(i)} = H\omega_i\alpha^{-1}$ so that $\alpha_i = \omega_{\alpha(i)}\alpha\omega_i^{-1} \in H$ for all $\alpha \in G, i \in H\backslash G$.

For $\beta \in G$, the map $j \mapsto \alpha(j)$ carries bijectively $N(\beta)$ onto $N(\alpha\beta\alpha^{-1})$. For every $j \in N(\beta)$, we have the functor

$$\varphi_{\alpha_j} : \mathcal{C}_{\omega_j\beta\omega_j^{-1}} \to \mathcal{C}_{\omega_{\alpha(j)}\alpha\beta\alpha^{-1}(\omega_{\alpha(j)})^{-1}}.$$

Given an object $V = (\beta, B, \{V_j\}_{j\in B})$ of $\widehat{\mathcal{C}}_\beta$, we apply $\{\varphi_{\alpha_j}\}_{j\in B}$ coordinate-wise to obtain an object $\widetilde{\varphi}_\alpha(V) = (\alpha\beta\alpha^{-1}, \alpha(B), \{\varphi_{\alpha_j}(V_j)\}_{j\in B})$ of $\widehat{\mathcal{C}}_{\alpha\beta\alpha^{-1}}$. More precisely,

$$\widetilde{\varphi}_\alpha(V) = (\alpha\beta\alpha^{-1}, \alpha(B), \{\varphi_{\alpha_{\alpha^{-1}(i)}}(V_{\alpha^{-1}(i)})\}_{i\in\alpha(B)}).$$

Clearly, $|\widetilde{\varphi}_\alpha(V)| = \alpha(|V|)$. The action of $\widetilde{\varphi}_\alpha$ on a morphism $f\colon V \to V'$ is given in the coordinates by the formula

$$\widetilde{\varphi}_\alpha(f) = \{\varphi_{\alpha_{\alpha^{-1}(i)}}(f_{\alpha^{-1}(i)}\colon V_{\alpha^{-1}(i)} \to V'_{\alpha^{-1}(i)})\}_{i\in\alpha(|V|)\cap\alpha(|V'|)}.$$

The functor $\widetilde{\varphi}_\alpha$ preserves the structural morphisms in $\widehat{\mathcal{C}}$ because by assumptions the functors $\{\varphi_{\alpha_j}\}_j$ preserve the structural morphisms in $\mathcal{C}$. The pair $(\widehat{\mathcal{C}}, \widetilde{\varphi})$ is a crossed G-category. Its isomorphism class does not depend on the choice of $\omega_i \in i$.

3.3 Transfer of braided and ribbon *H*-categories. If $\mathcal{C}$ is a braided crossed H-category, then its transfer $\widehat{\mathcal{C}}$ acquires a structure of a braided G-category as follows. Consider any objects $U = (\alpha, A, \{U_i\}_{i\in A})$ and $V = (\beta, B, \{V_j\}_{j\in B})$ of $\widehat{\mathcal{C}}$. By definition,

$$\begin{aligned} U \otimes V &= (\alpha\beta, A\cap B, \{U_i \otimes V_i\}_{i\in A\cap B}), \\ {}^UV &= \widetilde{\varphi}_\alpha(V) = (\alpha\beta\alpha^{-1}, \alpha(B), \{\varphi_{\alpha_{\alpha^{-1}(i)}}(V_{\alpha^{-1}(i)})\}_{i\in\alpha(B)}), \\ {}^UV \otimes U &= (\alpha\beta, \alpha(B)\cap A, \{\varphi_{\alpha_{\alpha^{-1}(i)}}(V_{\alpha^{-1}(i)}) \otimes U_i\}_{i\in\alpha(B)\cap A}). \end{aligned}$$

The latter expression simplifies if we observe that for any $i \in N(\alpha)$, we have $H\omega_i = H\omega_i\alpha^{-1}$ and therefore $\alpha(i) = i$. This implies that the map $i \mapsto \alpha(i)\colon H\backslash G \to H\backslash G$ is the identity on A. Hence, $\alpha(B)\cap A = B\cap A = A\cap B$ and

$${}^UV \otimes U = (\alpha\beta, A\cap B, \{\varphi_{\alpha_i}(V_i) \otimes U_i\}_{i\in A\cap B}).$$

We define a morphism $c_{U,V}\colon U\otimes V \to {}^UV \otimes U$ by its coordinates

$$(c_{U,V})_i = c_{U_i,V_i} : U_i \otimes V_i \to \varphi_{\alpha_i}(V_i) \otimes U_i,$$

where i runs over $A \cap B$ and c_{U_i,V_i} is the braiding in $\mathcal{C}$. Note that

$$\varphi_{\alpha_i}(V_i) = \varphi_{\omega_i \alpha \omega_i^{-1}}(V_i) = {}^{U_i}(V_i).$$

The morphisms $\{c_{U,V}\}_{U,V}$ satisfy the axioms of a braiding in the crossed G-category $\widehat{\mathcal{C}}$; this is easily verified coordinate-wise.

Similarly, if $\mathcal{C}$ is a ribbon crossed H-category, then its transfer $\widehat{\mathcal{C}}$ has a structure of a ribbon G-category. For an object $U = (\alpha, A, \{U_i\}_{i\in A}) \in \widehat{\mathcal{C}}$, we have ${}^{U}U = (\alpha, A, \{\varphi_{\alpha_i}(U_i)\}_{i\in A})$. Here $\alpha_i = \omega_i \alpha \omega_i^{-1}$ so that $\varphi_{\alpha_i}(U_i) = {}^{U_i}(U_i)$. The twist $\theta_U : U \to {}^{U}U$ is defined by its coordinates $(\theta_U)_i = \theta_{U_i} : U_i \to {}^{U_i}(U_i)$, where i runs over A. The morphisms $\{\theta_U\}_U$ satisfy the axioms of a twist in the braided G-category $\widehat{\mathcal{C}}$.

The algebra of endomorphisms of the unit object of $\widehat{\mathcal{C}}$ is computed by

$$\mathrm{End}_{\widehat{\mathcal{C}}}(\mathbb{1}_{\widehat{\mathcal{C}}}) = (\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}))^{[G:\,H]}.$$

Therefore, unfortunately, the transfer of an H-category cannot yield a modular G-category except in the uninteresting case $H = G$.

VIII.4 Quasi-abelian cohomology of groups

4.1 Quasi-abelian cohomology. Let $a = \{a_{\alpha,\beta,\gamma} \in K^*\}_{\alpha,\beta,\gamma\in G}$ be a 3-cocycle of a group G with values in K^* and $c = \{c_{\alpha,\beta} \in K^*\}_{\alpha,\beta\in G}$ be a family of elements of K^*. Equations (2.6.a), (2.6.c), (2.6.e), (2.6.f) in Chapter VI, on the pair (a, c) are equivalent to the equations introduced by C. Ospel [Osp1], [Osp2] from a different viewpoint (in his notation $c_{\alpha,\beta} = \Omega_{\beta,\alpha}$ and $a_{\alpha,\beta,\gamma} = f(\alpha,\beta,\gamma)$). Following Ospel, we call such pairs (a, c) satisfying (2.6.a), (2.6.c), (2.6.e), (2.6.f) *quasi-abelian* 3*-cocycles* on G. Examples of quasi-abelian 3-cocycles are provided by pairs (a, c) where $a = 1$ and c is an arbitrary bilinear form $H_1(G) \times H_1(G) \to K^*$, $(\alpha,\beta) \mapsto c_{\alpha,\beta}$.

Quasi-abelian 3-cocycles (a, c) on G form a commutative group under pointwise multiplication with unit $(a, c) = (1, 1)$. This group contains the coboundaries of the conjugation invariant 2-cochains. A conjugation invariant 2-cochain is a map $G \times G \to K^*$, $(\alpha,\beta) \mapsto \eta_{\alpha,\beta}$ such that $\eta_{\delta\alpha\delta^{-1},\delta\beta\delta^{-1}} = \eta_{\alpha,\beta}$ for all $\alpha,\beta,\delta \in G$. Its coboundary is defined by

$$a_{\alpha,\beta,\gamma} = \eta_{\alpha,\beta}\, \eta_{\alpha\beta,\gamma} \eta^{-1}_{\alpha,\beta\gamma} \eta^{-1}_{\beta,\gamma}, \quad c_{\alpha,\beta} = \eta_{\alpha,\beta}\, \eta^{-1}_{\beta,\alpha}$$

for all $\alpha,\beta,\gamma \in G$. A direct computation shows that this pair (a, c) is a quasi-abelian 3-cocycle. The quotient of the group of quasi-abelian 3-cocycles by the subgroup of coboundaries is the group $H^3_{qa}(G; K^*)$ of quasi-abelian cohomology of G. Involution (2.6.i) (Chapter VI), transforms the subgroup of coboundaries into itself and defines an involution on $H^3_{qa}(G; K^*)$. Forgetting c we obtain a homomorphism to the standard cohomology $H^3_{qa}(G; K^*) \to H^3(G; K^*)$.

The constructions of Section VI.2.6 associate with any quasi-abelian 3-cocycle of G and a conjugation invariant family $\{b_\alpha \in K^*\}_{\alpha\in G}$ a certain braided crossed G-category $\mathcal{C}$. A twist in $\mathcal{C}$ is determined by a family $\{\theta_\alpha \in K^*\}_{\alpha\in G}$ satisfying equations (2.6.g), (2.6.h) in Chapter VI. The next lemma describes all twists in $\mathcal{C}$.

4.2 Lemma. *Let (a, c) be a quasi-abelian 3-cocycle of G. Then $\{\theta_\alpha = c_{\alpha,\alpha}\}_{\alpha\in G}$ satisfies (2.6.g), (2.6.h) (Chapter VI). A general solution to (2.6.g), (2.6.h) is the product of this solution with a group homomorphism $G \to \{k \in K \mid k^2 = 1\} \subset K^*$.*

Proof. It is obvious that any two solutions to (2.6.g) (with given c) are obtained from each other by multiplication by a group homomorphism $G \to K^*$. The condition $\theta_{\alpha^{-1}} = \theta_\alpha$ holds if and only if this homomorphism takes values in the group $\{k \in K \mid k^2 = 1\}$.

Let us prove that $\{\theta_\alpha = c_{\alpha,\alpha}\}_{\alpha\in G}$ satisfies (2.6.g). A direct computation using consecutively (2.6.e), (2.6.f), (2.6.a) (Chapter VI) yields

$$c_{\alpha\beta,\alpha\beta} = c_{\alpha,\alpha}c_{\beta,\beta}c_{\alpha,\beta}c_{\beta,\alpha}(a_{\alpha\beta,\alpha,\beta}a_{\alpha,\beta,\alpha\beta})^{-1}a_{\alpha\beta\alpha\beta^{-1}\alpha^{-1},\alpha\beta,\beta}a_{\beta\alpha\beta^{-1},\beta,\alpha}a_{\alpha\beta\alpha^{-1},\alpha,\beta}.$$

Formula (2.6.g) is equivalent to the fact that the product on the second line is equal to 1. This equality can be deduced from (1.3.a) (Chapter VI) by applying the substitution $\alpha \mapsto \beta\alpha\beta^{-1}, \gamma \mapsto \alpha, \delta \mapsto \alpha^{-1}\beta\alpha$ and using (2.6.a).

It remains to check that $c_{\alpha^{-1},\alpha^{-1}} = c_{\alpha,\alpha}$ for all α. Applying (2.6.e) to $\beta = \gamma = 1$ we obtain $c_{\alpha,1} = a_{\alpha,1,1}\, a_{1,1,\alpha}$. Applying (2.6.e) to $\beta = \alpha, \gamma = \alpha^{-1}$ we obtain

$$c_{\alpha,\alpha^{-1}} = c_{\alpha,\alpha}^{-1}\, c_{\alpha,1}\, a_{\alpha,\alpha^{-1},\alpha} = c_{\alpha,\alpha}^{-1}\, a_{\alpha,1,1}\, a_{1,1,\alpha}\, a_{\alpha,\alpha^{-1},\alpha}.$$

Similarly, substituting $\beta = \delta = 1$, $\gamma = \alpha$ and $\beta = \alpha$, $\gamma = \delta = \alpha^{-1}$ in (2.6.f) we obtain $c_{1,\alpha} = (a_{\alpha,1,1}\, a_{1,1,\alpha})^{-1}$ and

$$c_{\alpha^{-1},\alpha^{-1}} = c_{1,\alpha^{-1}}\,(c_{\alpha,\alpha^{-1}}\, a_{\alpha^{-1},\alpha,\alpha^{-1}})^{-1} = (c_{\alpha,\alpha^{-1}}\, a_{\alpha^{-1},1,1}\, a_{1,1,\alpha^{-1}}\, a_{\alpha^{-1},\alpha,\alpha^{-1}})^{-1}.$$

Substituting here the expressions for $c_{\alpha,\alpha^{-1}}$ obtained above and using that the right-hand sides of (1.3.b) and (1.3.c) in Chapter VI, are equal we obtain $c_{\alpha^{-1},\alpha^{-1}} = c_{\alpha,\alpha}$. □

VIII.5 Remarks on group-algebras

5.1 Push-forward for group-algebras. Consider a group homomorphism $q\colon G' \to G$. We can push forward any G'-algebra L' along q to obtain a G-algebra $L = q_*(L')$. By definition, for $\alpha \in G$,

$$L_\alpha = \bigoplus_{u\in q^{-1}(\alpha)} L'_u.$$

Multiplication in L is induced by multiplication in L' in the obvious way. Note that $L = L'$ as algebras because

$$L = \bigoplus_{\alpha\in G} L_\alpha = \bigoplus_{u\in G'} L'_u = L'.$$

Suppose that L' is a crossed G-algebra with automorphisms $\{\varphi'_u\}_{u\in G'}$. Suppose also that the homomorphism $q\colon G' \to G$ is surjective and its kernel $\Gamma = \operatorname{Ker} q$ is finite and is contained both in the center of G' and in $\operatorname{Ker}\varphi'$. Then φ' induces an action φ of G on L by $\varphi_\alpha = \varphi'_u$ for any $\alpha \in G$ and $u \in q^{-1}(\alpha)$. Here we use the equality $L = L'$ above. We claim that L is a crossed G-algebra. Axioms (3.1.1)–(3.1.3) in Chapter II for φ directly follow from the corresponding axioms for φ'. Let us check Axiom (3.1.4). Let $\alpha, \beta \in G$ and $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$. Note that for $u \in q^{-1}(\alpha)$, $v \in q^{-1}(\beta)$, the commutator $uvu^{-1}v^{-1}$ does not depend on the choice of u and v. Denote this commutator by $w_{\alpha,\beta}$. To check equality (3.1.a) in Chapter II it suffices to consider the case where $c \in L'_w$ with $w \in q^{-1}(\alpha\beta\alpha^{-1}\beta^{-1})$. The homomorphism

$$\mu_c\,\varphi_\beta = \mu_c\,\varphi'_v\colon L_\alpha \to L_\alpha$$

carries a direct summand L'_u of L_α (with $u \in q^{-1}(\alpha)$) into $L'_{wvuv^{-1}}$. Therefore

$$\begin{aligned}\operatorname{Tr}(\mu_c\,\varphi_\beta\colon L_\alpha \to L_\alpha) &= \sum_{\substack{u\in q^{-1}(\alpha)\\ wvuv^{-1}=u}} \operatorname{Tr}(\mu_c\,\varphi'_v\colon L'_u \to L'_u)\\ &= \begin{cases}\sum_{u\in q^{-1}(\alpha)} \operatorname{Tr}(\mu_c\,\varphi'_v\colon L'_u \to L'_u) & \text{if } w = w_{\alpha,\beta},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The assumption $\varphi'(\Gamma) = 1$ implies that the trace $\operatorname{Tr}(\mu_c\,\varphi'_v\colon L'_u \to L'_u)$ does not depend on the choice of v in $q^{-1}(\beta)$. The same argument together with formula (3.1.a) in Chapter II for L' show that in the case $w = w_{\alpha,\beta}$, this trace does not depend on the choice of u in $q^{-1}(\alpha)$. Hence, for any $u \in q^{-1}(\alpha)$ and $v \in q^{-1}(\beta)$,

$$\operatorname{Tr}(\mu_c\,\varphi_\beta\colon L_\alpha \to L_\alpha) = \begin{cases}|\Gamma|\ \operatorname{Tr}(\mu_c\,\varphi'_v\colon L'_u \to L'_u) & \text{if } w = w_{\alpha,\beta},\\ 0 & \text{otherwise.}\end{cases}$$

Similarly,

$$\begin{aligned}\operatorname{Tr}(\varphi_{\alpha^{-1}}\mu_c\colon L_\beta \to L_\beta) &= \sum_{\substack{v\in q^{-1}(\beta)\\ v=u^{-1}wvu}} \operatorname{Tr}(\varphi'_{u^{-1}}\mu_c\colon L'_v \to L'_v)\\ &= \begin{cases}|\Gamma|\ \operatorname{Tr}(\varphi'_{u^{-1}}\mu_c\colon L'_v \to L'_v) & \text{if } w = w_{\alpha,\beta},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now, Axiom (3.1.4) for L follows from Axiom (3.1.1) for L'.

If L' is a crossed Frobenius G'-algebra, then under the same assumptions on q, the push-forward L of L' is a crossed Frobenius G-algebra.

As an example, consider the standard structure of a crossed G'-algebra on the group ring $K[G']$. Let G be the quotient of G' by a finite subgroup Γ of the center of G'. By the construction above, we obtain a structure of a crossed Frobenius G-algebra on the same ring $L = K[G']$. Here G acts on L by conjugations and the inner product

η is given by $\eta(u, v) = \delta^1_{uv}$ for $u, v \in G'$, where δ is the Kronecker delta. By definition, $L_\alpha = u\,K[\Gamma]$, for any lift $u \in G'$ of $\alpha \in G$. It is worthwhile to note natural deformations of η in the class of inner products. Namely, choose a non-degenerate symmetric bilinear form $\rho\colon L_1 \times L_1 \to K$ on $L_1 = K[\Gamma]$ such that the pair (L_1, ρ) is a Frobenius algebra. (There are many such forms as it is easy to see for cyclic Γ.) We define an inner product $\eta_\rho\colon L \otimes L \to K$ by $\eta_\rho(L_\alpha \otimes L_\beta) = \delta^1_{\alpha\beta}\,\rho(ab, 1_L)$ for any $a \in L_\alpha, b \in L_\beta$ with $\alpha, \beta \in G$. Then (L, η_ρ) is a crossed Frobenius G-algebra. If K has a ring involution $K \to K, k \mapsto \bar{k}$ such that $\rho(u^{-1}, v^{-1}) = \overline{\rho(u, v)}$ for any $u, v \in \Gamma$, then the antilinear involution in $L = K[G']$ carrying each $u \in G'$ to u^{-1} is Hermitian.

5.2 Crossed group-algebras from algebra automorphisms. Let us consider a crossed Frobenius G-algebra L such that $L_\alpha = 0$ for all $\alpha \neq 1$. Axioms (2.1.1)–(2.1.3), p. 24, and (3.1.1)–(3.1.3), p. 25, amount to saying that L_1 is a commutative Frobenius algebra with an action of G by algebra automorphisms preserving the inner product. We call such L_1 a *G-equivariant Frobenius algebra*. Axiom (3.1.4) means that the action of G is traceless in the sense that $\mathrm{Tr}(\mu_c\,\varphi_\alpha\colon L_1 \to L_1) = 0$ for all $c \in L_1$ and $\alpha \neq 1$. Thus, any traceless G-equivariant Frobenius algebra extends to a crossed Frobenius G-algebra such that $L_\alpha = 0$ for $\alpha \neq 1$. Note that the tensor product $L_1 \otimes L'_1$ of two G-equivariant Frobenius algebras is a G-equivariant Frobenius algebra. If L_1 or L'_1 is traceless, then so is $L_1 \otimes L'_1$.

Equivariant Frobenius algebras naturally arise in the study of groups of homeomorphisms of manifolds. Consider a closed connected oriented even-dimensional manifold M and set $L_1 = L_1(M) = \bigoplus_{k \in 2\mathbb{Z}} H^k(M; \mathbb{Q})$. Multiplication in L_1 is the cup-product and the inner product on L is defined by $(a, b) \mapsto (a \cup b)([M])$. Clearly, L_1 is a commutative Frobenius algebra. Now, any group G of orientation preserving self-homeomorphisms of M acts on L_1 via induced homomorphisms. Clearly, $\mathrm{Tr}(\mu_c\,\alpha_*\colon L_1 \to L_1) = 0$ for all $\alpha \in G$ and $c \in H^k(M; \mathbb{Q})$ with $k > 0$. The only restriction arises from $c = 1 \in H^0(M; \mathbb{Q})$ and consists in the identity $\mathrm{Tr}(\alpha_*\colon L_1 \to L_1) = 0$ for all $\alpha \in G - \{1\}$. For instance, consider an orientation-reversing involution of the $2n$-dimensional sphere $j_n\colon S^{2n} \to S^{2n}$. It is clear that the action of j_n on $H^*(S^{2n}; \mathbb{Q})$ has a zero trace. Therefore for any orientation-reversing involution $j\colon M \to M$, the product $j_n \times j\colon S^{2n} \times M \to S^{2n} \times M$ induces an endomorphism of $L_1(S^{2n} \times M)$ with zero trace. For example, $j_n \times j_m$ is an endomorphism of $L_1(S^{2n} \times S^{2m})$ with zero trace. This construction yields examples of traceless $\mathbb{Z}/2\mathbb{Z}$-equivariant Frobenius algebras and hence of crossed Frobenius $\mathbb{Z}/2\mathbb{Z}$-algebras.

Here is another (related) construction of traceless G-equivariant Frobenius algebras. Consider a fixed point free group G of orientation preserving self-homeomorphisms of a manifold M as above. By the Lefschetz theorem, the super-trace of the induced action of G on $\bigoplus_k H^k(M; \mathbb{Q})$ is zero. If M has only even-dimensional cohomology, this gives a traceless G-equivariant Frobenius algebra. This example suggests a notion of a crossed super-G-algebra.

5.3 Further examples of Frobenius G-algebras. Let $\{V_s\}_{s\in S}$ be a family of K-modules labeled by elements of a finite set S. With every left action of G on S we associate a G-algebra $L = \bigoplus_{\alpha\in G} L_\alpha$, where

$$L_\alpha = \bigoplus_{s\in S} \mathrm{Hom}(V_s, V_{\alpha(s)}).$$

Each element $a \in L_\alpha$ is determined by its "coordinates"

$$\{a_s \in \mathrm{Hom}(V_s, V_{\alpha(s)})\}_{s\in S}.$$

The K-linear structure in L is coordinate-wise. For $a \in L_\alpha$ and $b \in L_\beta$, the product $ab \in L_{\alpha\beta}$ is defined in coordinates by $(ab)_s = a_{\beta(s)} b_s \in \mathrm{Hom}(V_s, V_{\alpha\beta(s)})$. This multiplication turns L into a G-algebra with unit $\bigoplus_s \mathrm{id}_{V_s}$.

Another interesting G-algebra $R = \bigoplus_{\alpha\in G} R_\alpha$ is defined by

$$R_\alpha = \bigotimes_{s\in S} \mathrm{Hom}(V_s, V_{\alpha(s)}).$$

The K-module R_α is additively generated by the vectors $a = \otimes_s a_s$ where $a_s \in \mathrm{Hom}(V_s, V_{\alpha(s)})$ for $s \in S$. If $b = \otimes_s b_s \in R_\beta$ with $b_s \in \mathrm{Hom}(V_s, V_{\beta(s)})$, then we set $ab = \otimes_s a_{\beta(s)} b_s \in R_{\alpha\beta}$. This extends by linearity to an associative multiplication on R with unit $\otimes_s \mathrm{id}_{V_s}$.

5.4 Exercises. 1. Verify that the G-algebra L defined in Section 5.3 is biangular if and only if either $V_s = 0$ for all $s \in S$ or $\mathrm{Dim}\, V_s \in K$ does not depend on s and is invertible in K. (Hint: use the following property of the trace. Let P and Q be free K-modules of finite rank. Then for any $\ell \in \mathrm{Hom}(Q, Q)$ and $\ell' \in \mathrm{Hom}(P, P)$, the trace of the endomorphism of $\mathrm{Hom}(P, Q)$ carrying $f \in \mathrm{Hom}(P, Q)$ to $\ell f \ell'$ is equal to $\mathrm{Tr}(\ell)\ \mathrm{Tr}(\ell')$.)

2. Verify that the G-algebra R defined in Section 5.3 is biangular if and only if is either $V_s = 0$ for all $s \in S$ or $\mathrm{Dim}\, V_s \in K$ is invertible in K for all $s \in S$.

3. Let G be a finite group. Any G-algebra $L = \bigoplus_{\alpha\in G} L_\alpha$ gives rise to another G-algebra $\widehat{L} = \bigoplus_{\alpha\in G} \widehat{L}_\alpha$, where $\widehat{L}_\alpha = \bigoplus_{\omega,\omega'\in G} L_{\omega\alpha\omega'}$. Multiplication in $\widehat{L}$ is defined by $L_{\omega_1\alpha_1\omega_1'} L_{\omega_2\alpha_2\omega_2'} = 0$ unless $\omega_1'\omega_2 = 1$ and then it is multiplication in L

$$L_{\omega_1\alpha_1\omega_1'} \times L_{\omega_2\alpha_2\omega_2'} \to L_{\omega_1\alpha_1\alpha_2\omega_2'} \subset \widehat{L}_{\alpha_1\alpha_2}.$$

Verify that if the underlying algebra of L is non-degenerate, then $\widehat{L}$ is biangular.

Appendix 1
Relative HQFTs

The definition of an HQFT can be generalized in various directions. For example, one can extend the class of manifolds and cobordisms by allowing manifolds with boundary and cobordisms with corners. One can also extend the class of maps to the target space X by allowing maps to carry the base points to a fixed subspace of X rather than to the base point. This leads us to a notion of a *relative HQFT*. Relative HQFTs include as a special case the so-called open-closed TQFTs introduced and studied by G. Moore and G. Segal [Mo], [MS]; see also [LP1], [LP2].

We outline here the definition of a relative HQFT and discuss the algebraic structures underlying certain 2-dimensional relative HQFTs.

1 Preliminaries. A manifold with boundary M is said to be ∂*-pointed* if all closed components of M and all components of ∂M are pointed. The distinguished points of the closed components of M and of the components of ∂M are called the *base points* of M.

Fix a connected CW-space X and its three CW-subspaces X^-, Y, Y^- such that $Y \supset Y^-$. Denote the tuple (X, X^-, Y, Y^-) by X_+. By an X_+*-manifold*, we mean a pair (a ∂-pointed compact oriented manifold M, a map $g\colon M \to X$) such that $g(\partial M) \subset Y$ and g carries the base points of the closed components of M to X^- and the base points of the components of ∂M to Y^-.

A disjoint union of X_+-manifolds is an X_+-manifold in the obvious way. An empty set is considered as an X_+-manifold of any given dimension. An X_+*-homeomorphism* of X_+-manifolds $f\colon (M, g) \to (M', g')$ is a base point preserving and orientation preserving diffeomorphism such that $g = g'f$.

By a $(d+1)$*-dimensional cobordism with corners*, we mean a tuple (W, M_0, M_1), where W is a compact oriented $(d+1)$-dimensional manifold, M_0, M_1 are disjoint ∂-pointed compact oriented d-dimensional submanifolds of ∂W, and the orientation of M_1 (resp. of M_0) is induced by that of W (resp. of $-W$). Set

$$\partial_\circ W = \partial W - (\mathrm{Int}\, M_0 \cup \mathrm{Int}\, M_1).$$

The d-dimensional cobordism $(\partial_\circ W, \partial M_0, \partial M_1)$ is the *boundary* of (W, M_0, M_1). Clearly, $\partial_\circ(\partial_\circ W) = \emptyset$.

An X_+*-cobordism* is a cobordism with corners (W, M_0, M_1) endowed with a map $g\colon W \to X$ such that $g(\partial_\circ W) \subset Y$ and g carries the base points of the closed components of M_0, M_1 to X^- and the base points of ∂M_0, ∂M_1 to Y^-. Both M_0 and M_1 are then X_+-manifolds with maps to X obtained by restricting g. *Gluings* and X_+*-homeomorphisms* of X_+-cobordisms are defined as in Section I.1.1 with the obvious changes.

2 Relative HQFTs. Let $X_+ = (X, X^-, Y, Y^-)$ be as above. For an integer $d \geq 0$, a *relative* $(d+1)$*-dimensional Homotopy Quantum Field Theory* (A, τ) *with target* X_+, or shorter a *relative* $(d+1)$*-dimensional* X_+*-HQFT* assigns a projective K-module of finite type A_M to any d-dimensional X_+-manifold M, a K-isomorphism $f_\#\colon A_M \to A_{M'}$ to any X_+-homeomorphism of d-dimensional X_+-manifolds $f\colon M \to M'$, and a K-homomorphism $\tau(W)\colon A_{M_0} \to A_{M_1}$ to any $(d+1)$-dimensional X_+-cobordism (W, M_0, M_1). These modules and homomorphisms should satisfy eight axioms obtained from the Axioms (1.2.1)–(1.2.8) in Chapter I by replacing X with X_+. In the analogue of Axiom (1.2.8) we consider only homotopies in the class of maps $W \to X$ constant on ∂W and carrying $\partial_\circ W$ to Y. For example, if $Y = \emptyset$ and X^- is a one-point set, then we recover the definition of an X-HQFT in Chapter I. In the remaining part of this Appendix, HQFTs in the sense of Chapter I are called *absolute HQFTs*.

All definitions and results of Chapter I can be generalized to relative HQFTs. In particular, any $\theta \in H^{d+1}(X, Y; K^*)$ determines a relative $(d+1)$-dimensional X_+-HQFT (A^θ, τ^θ). As in Section I.1.4, we can define a category $Q_{d+1}(X_+)$ of relative $(d+1)$-dimensional X_+-HQFTs. The construction $X_+ \mapsto Q_{d+1}(X_+)$ is contravariant with respect to maps of tuples $f\colon \underline{X}_+ \to X_+$ where $\underline{X}_+$ is a 4-tuple consisting of a connected CW-space $\underline{X}$ and its CW-subspaces $\underline{X}^-, \underline{Y} \supset \underline{Y}^-$ and f is a map $\underline{X} \to X$ such that $f(\underline{X}^-) \subset X^-$, $f(\underline{Y}) \subset Y$, and $f(\underline{Y}^-) \subset Y^-$. The induced functor $Q_{d+1}(X_+) \to Q_{d+1}(\underline{X}_+)$ is obtained by composing the maps to $\underline{X}_+$ with f. In particular, for any point $x \in X^-$, we can take $\underline{X}_+ = (X, x, \emptyset, \emptyset)$ and $f = \mathrm{id}_X\colon X \to X$ and obtain thus a functor $Q_{d+1}(X_+) \to Q_{d+1}(X)$ which associates with a relative X_+-HQFT the *induced absolute* X*-HQFT*.

The *boundary functor*

$$Q_d(Y, Y \cap Y^-, \emptyset, \emptyset) \to Q_{d+1}(X, X^-, Y, Y^-)$$

transforms a d-dimensional HQFT (A, τ) with target $(Y, Y \cap Y^-, \emptyset, \emptyset)$ into the $(d+1)$-dimensional X_+-HQFT $M \mapsto A_{\partial M}$, $W \mapsto \tau(\partial_\circ W)$.

3 The case $d = 1$. For $d = 1$, we enjoy a minor technical simplification due to the fact that connected 0-dimensional manifolds are points and thus are automatically pointed. To turn a 1-dimensional manifold M into a ∂-pointed manifold, it is enough to specify a base point on each closed component of M. The set of base points of M is then the union of ∂M with the set of base points of the closed components of M.

By an X_+*-curve*, we shall mean a 1-dimensional X_+-manifold, i.e., a pair (a ∂-pointed compact oriented 1-dimensional manifold M, a map $g\colon M \to X$) such that $g(\partial M) \subset Y^-$ and g carries the base points of the closed components of M to X^-. By an X_+*-surface*, we shall mean a 2-dimensional X_+-cobordism, i.e., a tuple $(W, M_0, M_1, g\colon W \to X)$, where W is a compact oriented surface, M_0, M_1 are disjoint ∂-pointed compact oriented 1-dimensional submanifolds of ∂W with orientation of M_1 (resp. of M_0) induced by that of W (resp. of $-W$), $g(\partial M_0 \cup \partial M_1) \subset Y^-$,

$g(\partial_\circ W) \subset Y$, and g carries the base points of the closed components of M_0, M_1 to X^-.

Relative 2-dimensional HQFTs with target $(X, x,\ X, x)$ were studied by Moore and Segal [Mo], [MS]. For a description of the algebraic structures underlying these HQFTs, see [MS], Theorem 7.

We analyze here the algebraic structures underlying a relative 2-dimensional HQFT (A, τ) with target $X_+ = (X, x, x, x)$, where $X = K(G, 1)$ and $x \in X$. (In the notation above, $X^- = Y = Y^- = \{x\}$.) First of all, restricting (A, τ) to closed 1-manifolds and cobordisms between them, we obtain the induced absolute X-HQFT. Let $(L = \bigoplus_{\alpha \in G} L_\alpha, \eta, \varphi)$ be its underlying crossed Frobenius G-algebra.

Further algebraic data is derived from (A, τ) as follows. Set $I = [0, 1]$ with orientation from 0 to 1 and represent any $\alpha \in G = \pi_1(X, x)$ by a loop $\tilde{\alpha}\colon I \to X$ beginning and ending at x. The pair $M_\alpha = (I, \tilde{\alpha})$ is an X_+-curve. Then $B_\alpha = A_{M_\alpha}$ is a projective K-module of finite type. As in Section III.1.1, this module does not depend on the choice of $\tilde{\alpha}$ in its homotopy class (up to canonical isomorphism).

The K-module $B = \bigoplus_{\alpha \in G} B_\alpha$ has an algebra structure as follows. Let $\alpha, \beta \in G$ be represented by loops $\tilde{\alpha}, \tilde{\beta}\colon I \to X$, respectively. Let 1_x be the constant loop $I \to \{x\}$. We define a path $\tilde{\alpha}\, 1_x\, \tilde{\beta}\colon I \to X$ by

$$\tilde{\alpha} 1_x \tilde{\beta}(t) = \begin{cases} \tilde{\alpha}(3t) & \text{if } t \in [0, 1/3], \\ x & \text{if } t \in [1/3, 2/3], \\ \tilde{\beta}(3t - 2) & \text{if } t \in [2/3, 1]. \end{cases}$$

The path $\tilde{\alpha} 1_x \tilde{\beta}$ is homotopic to the product path $\tilde{\alpha}\tilde{\beta}$. Any homotopy between these two paths determines an X_+-surface $V_{\alpha,\beta} = I^2 = I \times I$ with bottom base $M_0 = ([0, 1/3] \times 0) \cup ([2/3, 1] \times 0)$ and top base $M_1 = I \times 1$; see Figure 1.1, where the dotted lines represent the boundary of the cobordism (always carried to $\{x\}$ in this context). The homomorphism

$$\tau(V_{\alpha,\beta})\colon B_\alpha \otimes B_\beta = A_{M_0} \to A_{M_1} = B_{\alpha\beta} \tag{3.a}$$

does not depend on the choice of $\tilde{\alpha}, \tilde{\beta}$ in their homotopy classes or the choice of the homotopy between $\tilde{\alpha}\, 1_x\, \tilde{\beta}$ and $\tilde{\alpha}\, \tilde{\beta}$. The homomorphism (3.a) defines an associative multiplication in B. Generally speaking, this multiplication is non-commutative.

The constant mapping $I^2 \to \{x\}$ can be considered as an X_+-surface with empty bottom base and top base 1_x; see the left diagram in Figure 1.2. The associated homomorphism $K \to B_1$ carries $1 \in K$ to the two-sided unit of B. The same mapping $I^2 \to \{x\}$ can be viewed as an X_+-surface with bottom base 1_x and empty top base; see the right diagram in Figure 1.2. The associated homomorphism $f\colon B_1 \to K$ induces an inner product on B whose value on a pair $(a \in B_\alpha, b \in B_\beta)$ is zero if $\alpha\beta \neq 1$ and is $f(ab)$ if $\alpha\beta = 1$. In this way, B becomes a Frobenius G-algebra (exercise).

The last piece of the algebraic data underlying (A, τ) is a homomorphism $\iota\colon L \to B$ carrying L_α to B_α for all $\alpha \in G$. To define ι, consider the upper and lower half-circles

(oriented clockwise)

$$S^1_+ = \{z \in \mathbb{C} \mid |z| = 1,\ \mathrm{Im}(z) \geq 0\}, \quad S^1_- = \{z \in \mathbb{C} \mid |z| = 1,\ \mathrm{Im}(z) \leq 0\}.$$

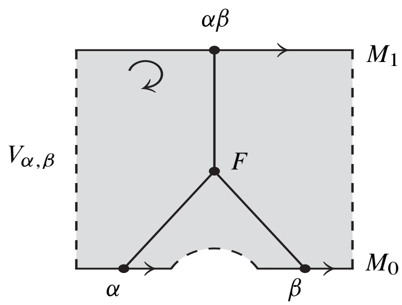


Figure 1.1. The X_+-surface $V_{\alpha,\beta}$.

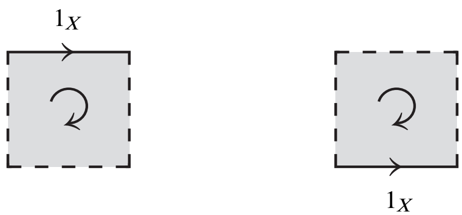


Figure 1.2. Two X_+-surfaces with bases $\emptyset$ and 1_X.

Represent $\alpha \in G$ by a loop $S^1 \to X$ carrying S^1_- to $\{x\}$. Let $\bar{\alpha}\colon S^1 \times I \to X$ be the composition of the projection $S^1 \times I \to S^1$ with this loop. We view the pair $W_\alpha = (S^1 \times I, \bar{\alpha})$ as an X_+-surface with bottom base $M_0 = S^1 \times \{0\}$ and top base $M_1 = S^1_+ \times \{1\}$; see Figure 1.3. Set

$$\iota|_{L_\alpha} = \tau(W_\alpha)\colon L_\alpha = A_{M_0} \to A_{M_1} = B_\alpha.$$

This defines a K-linear homomorphism $\iota\colon L \to B$.

The homomorphism $\iota\colon L \to B$ is an algebra homomorphism. A pictorial proof is given in Figure 1.4, where the left-hand side computes the composition of multiplication $L_\alpha \otimes L_\beta \to L_{\alpha\beta}$ with ι and the right-hand side computes the composition of $\iota\otimes\iota\colon L_\alpha \otimes L_\beta \to B_\alpha \otimes B_\beta$ with multiplication in B. The desired equality follows from the fact that the two X_+-surfaces in question are X_+-homeomorphic.

Note that

(i) for any $a \in L$ and $b \in B_\beta$ with $\beta \in G$, we have $\iota(\varphi_\beta(a))\, b = b\, \iota(a)$;

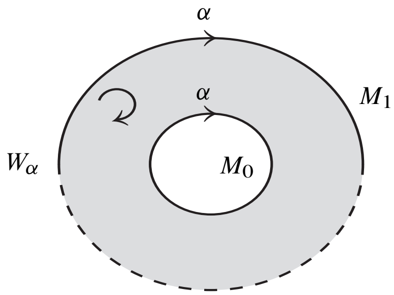


Figure 1.3. The X_+-surface W_α.

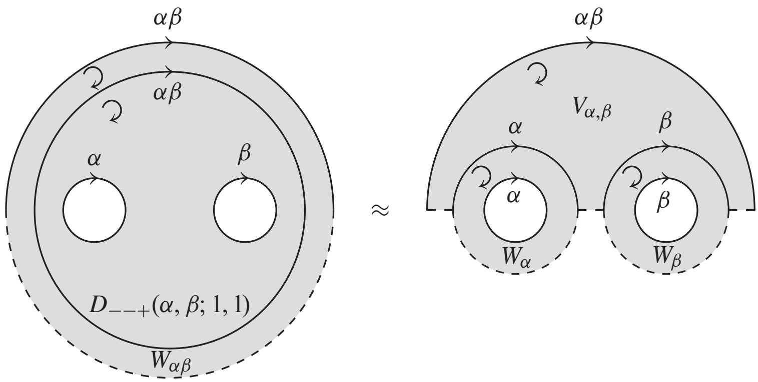


Figure 1.4. Two homeomorphic X_+-surfaces.

(ii) if $\iota^* : B \to L$ is adjoint to ι with respect to the inner products in B and L, then for any $\alpha \in G$,

$$\iota\, \varphi_\alpha\, \iota^* = \psi_\alpha : B \to B, \tag{3.b}$$

where ψ_α is the K-linear endomorphism of B defined in Section IV.1.2.

For $G = 1$, the properties (i) and (ii) are verified in [Mo], [MS], where (3.b) is called the *Cardy condition*. The case of an arbitrary G is similar.

These results suggest the following definition. A *Frobenius G-triple* is a triple consisting of a crossed Frobenius G-algebra L, a Frobenius G-algebra B, and a homomorphism of G-algebras $\iota : L \to B$ satisfying (i) and (ii). The Frobenius G-triple (L, B, ι) derived above from a relative 2-dimensional X_+-HQFT (A, τ) with $X_+ = (X, x, x, x)$ is called the *underlying Frobenius G-triple* of (A, τ). It is clear that the following diagram commutes:

$$\begin{array}{ccc} \{\text{relative } X_+\text{-HQFTs}\} & \longrightarrow & \{\text{absolute } X\text{-HQFTs}\} \\ \downarrow & & \downarrow \\ \{\text{Frobenius } G\text{-triples}\} & \longrightarrow & \{\text{crossed Frobenius } G\text{-algebras}\}. \end{array}$$

Here the vertical arrows associate with HQFTs their underlying algebraic objects. The upper horizontal arrow associates with a relative HQFT the induced absolute HQFT. The bottom horizontal arrow associates with a Frobenius G-triple (L, B, ι) the crossed Frobenius G-algebra L.

As we know, the right vertical arrow in this diagram is an equivalence of categories. It is natural to believe that the left vertical arrow also is an equivalence of categories. In particular, this would imply that every Frobenius G-triple underlies a relative 2-dimensional X_+-HQFT unique up to isomorphism. The cut and paste techniques of Chapter III should be sufficient to verify these assertions.

4 Relative HQFTs from biangular G-algebras. Let $X_+ = (X, x, x, x)$, where $X = K(G, 1)$ and $x \in X$. An arbitrary biangular G-algebra B gives rise to a relative 2-dimensional X_+-HQFT by extension of the state sum construction of Section IV.3.3 to X_+-curves and X_+-surfaces. One follows the same three steps. First, one introduces trivialized X_+-curves repeating the definitions of Section IV.3.3 word for word. For each trivialized X_+-curve M, one defines a K-module A°_M exactly as there. Secondly, one defines for each X_+-surface W with trivialized bases M_0 and M_1 a K-homomorphism $\tau^\circ(W)\colon A^\circ_{M_0} \to A^\circ_{M_1}$. The definition proceeds as in Section IV.3.3 with one difference: instead of formula (3.3.a) in Chapter IV one uses the formula

$$\otimes_{e \subset M_0 \cup M_1} B(\Delta_e, e, g) \to K, \quad a \mapsto D_g(a \otimes \eta_g^- \otimes 1_g),$$

where

$$1_g = \otimes_{e \subset \partial_\circ W}\, 1_B \in \otimes_{e \subset \partial_\circ W} B(\Delta_e, e, g).$$

(Note that if $e \subset \partial_\circ W$, then $g_e = 1$ and $B(\Delta_e, e, g) = B_1$.) The rest of the construction proceeds as in Section IV.3.3 and produces a relative 2-dimensional X_+-HQFT (A_B, τ_B). By the very definition, this relative X_+-HQFT induces the absolute X-HQFT constructed in Section IV.3. Note that to prove the appropriate extension of Lemma IV.3.3.1 to this setting, one needs to consider skeletons of pairs (a compact oriented surface W, a pointed compact 1-dimensional submanifold M of ∂W). Such skeletons are defined as skeletons of W in Section IV.4.1 but replacing in the last condition the words "each component N of ∂W" with the words "each component N of M". The rest of Section IV.4 easily adapts to skeletons of pairs and yields a proof of the extended Lemma IV.3.3.1.

4.1 Theorem. *Let B be a biangular G-algebra, and let $L \subset B$ be its G-center. Then the triple $(L, B$, the inclusion $\iota\colon L \hookrightarrow B)$ is a Frobenius G-triple. It is isomorphic to the Frobenius G-triple underlying the relative X_+-HQFT (A_B, τ_B).*

Proof. We check the first claim. Condition (i) in the definition of a Frobenius G-triple follows from formula (1.3.e) in Chapter IV, with $\alpha = \beta$. To verify Condition (ii) note that here $\iota^* = \psi_1 : B \to L$. Therefore $\iota \varphi_\alpha \iota^* = \iota \psi_\alpha \psi_1 = \psi_\alpha$ by formula (1.3.c) (Chapter IV), where we substitute $\beta = 1$ and use that $\psi_\alpha(1_B) = 1_B$ by (2.1.a) (Chapter IV).

By Section 3, the relative X_+-HQFT $(A, \tau) = (A_B, \tau_B)$ gives rise to a Frobenius G-algebra $B' = \bigoplus_{\alpha \in G} B'_\alpha$, a crossed Frobenius G-algebra L', and a homomorphism of G-algebras $\iota' : L' \to B'$. By Theorem IV.3.3.3, we have $L' = L$.

We compute B'_α for $\alpha \in G = \pi_1(X, x)$. Pick a loop $\tilde{\alpha} : I \to X$ based at x and representing α. We provide the X_+-curve $M = (I, \tilde{\alpha})$ with the trivialization formed by the vertices $\{0, 1\}$, one edge, and the G-system assigning α to this edge oriented from 0 to 1. By definition, $A^\circ_M = B_\alpha$ and A_M is the image of the projector $\tau^\circ(I^2) : A^\circ_M \to A^\circ_M$ associated with the square I^2 endowed with the map $(s, t) \mapsto \tilde{\alpha}(s) \in X$, where $s, t \in I$. Here the horizontal sides $I \times \{0\}$ and $I \times \{1\}$ of I^2 are identified with M in the obvious way, and $\partial_\circ(I^2)$ is the union of the vertical sides of I^2. To compute $\tau^\circ(I^2)$, we can use the standard CW-decomposition of the square formed by its four vertices, four sides, and the only face. A direct computation from the definitions shows that $\tau^\circ(I^2) = \mathrm{id}$ is the identity endomorphism of $A^\circ_M = B_\alpha$. Therefore $A_M = B_\alpha$ and $B'_\alpha = A_M = B_\alpha$. This proves that $B' = B$ as G-graded K-modules. That $B' = B$ as algebras follows from the definition of multiplication in B' given in Section 3 using the obvious CW-decomposition of $V_{\alpha,\beta}$ formed by six vertices and six edges (all lying on the boundary) and one face. Indeed, a direct computation shows that $\tau(V_{\alpha,\beta}) = \tau^\circ(V_{\alpha,\beta}) : B_\alpha \otimes B_\beta \to B_{\alpha\beta}$ is multiplication in B for all $\alpha, \beta \in G$.

Similar computations apply to the X_+-surfaces used in Section 3 to define the unit and the inner product in B'. These computations show that the unit and the inner product in B' are the same as in B.

To compute $\iota' : L' = L \to B = B'$, one can use the CW-decomposition of the cylinder $S^1 \times I$ formed by three vertices $(1, 0)$, $(-1, 1)$, $(1, 1)$, four edges $S^1 \times \{0\}$, $S^1_+ \times \{1\}$, $S^1_- \times \{1\}$, $\{1\} \times I$, and one face. This gives that $\iota' = \psi_1 \circ \iota$, where ι is the inclusion $L \hookrightarrow B$. As we know, $\psi_1|_L = \mathrm{id}_L$, so that $\iota' = \iota$. □

5 Other targets. One can similarly analyze the algebraic structures underlying a relative 2-dimensional HQFT (A, τ) with an arbitrary target $X_+ = (X, X^-, Y, Y^-)$. We point out a few interesting features first observed in [Mo] in the case $G = 1$. Any path in X from $x \in Y^-$ to $y \in Y^-$ can be viewed as an X_+-manifold. The HQFT (A, τ) associates with this X_+-manifold a projective K-module of finite type C_α depending only on the homotopy class α of the path. Set $\mathrm{Hom}(x, y) = \bigoplus_\alpha C_\alpha$, where α runs over the homotopy classes of paths from x to y. This yields a category C whose objects are points of Y^-. The morphisms $x \to y$ in C are elements of $\mathrm{Hom}(x, y)$. Composition in C is defined similarly to multiplication in the algebra B in Section 3. For all $x \in Y^-$, we have a "trace" function $f_x : \mathrm{Hom}(x, x) \to K$

obtained by composing the projection $\mathrm{Hom}(x,x) \to C_{[1_x]}$ with the homomorphism $C_{[1_x]} \to K$ determined by the right X_+-cobordism in Figure 1.2. One can show that for any homotopy class of paths α from $x \in Y^-$ to $y \in Y^-$, the bilinear pairing $(a,b) \to f_x(ab)\colon C_\alpha \times C_{\alpha^{-1}} \to K$ is non-degenerate and $f_x(ab) = f_y(ba)$. It would be interesting to study such "Frobenius categories" over groupoids.

6 Exercise. Let X be a connected aspherical CW-space. Let X^-, Y, Y^- be CW-subspaces of X such that $Y \supset Y^-$ and X^- is connected. Prove that for any $x \in X^-$ and $d \geq 0$, the functor $\mathcal{Q}_{d+1}(X, X^-, Y, Y^-) \to \mathcal{Q}_{d+1}(X, x, Y, Y^-)$ induced by the identity map $X \to X$ is an equivalence of categories. Hint: use the constructions of Section I.3.3.

Appendix 2

State sum invariants of 3-dimensional G-manifolds

1 Spherical categories. Let $\mathcal{C}$ be a K-additive monoidal category with structural morphisms a, l, r, b, d as in Section VI.1. Using these morphisms, one can define for any objects $V, W \in \mathcal{C}$ a canonical isomorphism $\gamma_{V,W}: V^* \otimes W^* \to (W \otimes V)^*$ (see, for instance, [Tu2], p. 31). In particular, if $\mathcal{C}$ is strict, then

$$\gamma_{V,W} = (d_V \otimes \mathrm{id}_{(W\otimes V)^*})(\mathrm{id}_{V^*} \otimes d_W \otimes \mathrm{id}_V \otimes \mathrm{id}_{(W\otimes V)^*})(\mathrm{id}_{V^*} \otimes \mathrm{id}_{W^*} \otimes b_{W\otimes V})$$

and

$$\gamma_{V,W}^{-1} = (d_{W\otimes V} \otimes \mathrm{id}_{V^*} \otimes \mathrm{id}_{W^*})(\mathrm{id}_{(W\otimes V)^*} \otimes \mathrm{id}_W \otimes b_V \otimes \mathrm{id}_{W^*})(\mathrm{id}_{(W\otimes V)^*} \otimes b_W).$$

Recall also that the duality b, d in $\mathcal{C}$ defines a contravariant functor $*: \mathcal{C} \to \mathcal{C}$ (see Section VI.2.4. In the next paragraph we use the morphism

$$\nu = (r_{\mathbb{1}^*})^{-1}\, b_{\mathbb{1}}: \mathbb{1} \to \mathbb{1}^*.$$

It is easy to deduce from the axioms of a monoidal category that ν is invertible.

The data making $\mathcal{C}$ *pivotal* in the sense of [FY], [BW3] is a system of morphisms $\{\tau_V : V \to V^{**}\}_{V\in\mathcal{C}}$ satisfying the following two conditions:

(1) τ is a natural transformation $\mathrm{id}_{\mathcal{C}} \to **$, i.e., for any morphism $f: V \to W$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \tau_V}\downarrow & & \downarrow{\scriptstyle \tau_W} \\ V^{**} & \xrightarrow{\;f^{**}\;} & W^{**}; \end{array}$$

(2) τ is monoidal, i.e., $\tau_{\mathbb{1}} = (\nu^{-1})^*\nu$ and for any objects $V, W \in \mathcal{C}$, the following diagram is commutative:

$$\begin{array}{ccc} V \otimes W & \xrightarrow{\;\tau_{V\otimes W}\;} & (V \otimes W)^{**} \\ {\scriptstyle \tau_V\otimes\tau_W}\downarrow & & \downarrow{\scriptstyle \gamma^*_{W,V}} \\ V^{**} \otimes W^{**} & \xrightarrow{\;\gamma_{V^*,W^*}\;} & (W^* \otimes V^*)^*. \end{array}$$

These conditions imply that $\tau_V: V \to V^{**}$ is invertible for all V (see [BV], Lemma 3.4) and $\tau_{V^*} = (\tau_V^{-1})^*: V^* \to V^{***}$ for any object $V \in \mathcal{C}$.

For an endomorphism f of an object V of a pivotal category $\mathcal{C}$, one defines a trace $\mathrm{tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ by

$$\mathrm{tr}(f) = d_{V^*}(\tau_V f \otimes \mathrm{id}_{V^*})\, b_V: \mathbb{1} \to \mathbb{1}.$$

We have $\mathrm{tr}(\mathrm{id}_{\mathbb{1}}) = \mathrm{id}_{\mathbb{1}}$ and $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ for any morphisms $f: V \to W$ and $g: W \to V$. Note that $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative K-algebra with multiplication defined by composition of morphisms and with unit $\mathrm{id}_{\mathbb{1}}$.

Following [BW3], we call a pivotal category $\mathcal{C}$ *spherical* if $\mathrm{tr}(f) = \mathrm{tr}(f^*)$ for any endomorphism f of an object of $\mathcal{C}$.

2 Remarks. 1. A pivotal category $\mathcal{C}$ has a right duality $\tilde{b}$, $\tilde{d}$ defined by

$$\tilde{b}_V = (\mathrm{id}_{V^*} \otimes \tau_V^{-1})\, b_{V^*}: \mathbb{1} \to V^* \otimes V,$$
$$\tilde{d}_V = d_{V^*}(\tau_V \otimes \mathrm{id}_{V^*}): V \otimes V^* \to \mathbb{1}.$$

One can check that the contravariant duality functor $*$ defined in Section VI.2.4 using b, d coincides with the similar functor derived from $\tilde{b}$, $\tilde{d}$. The isomorphisms $V^* \otimes W^* \to (W \otimes V)^*$ defined using b, d and $\tilde{b}$, $\tilde{d}$ also coincide, and

$$\nu = r_{\mathbb{1}^*}^{-1}\, b_{\mathbb{1}} = l_{\mathbb{1}^*}^{-1}\, \tilde{b}_{\mathbb{1}}: \mathbb{1} \to \mathbb{1}^*.$$

2. Our definition of a pivotal category differs from but is essentially equivalent to the one in [BW3]. Barrett and Westbury define a pivotal category as a monoidal category $\mathcal{C}$ equipped with a contravariant functor $*: \mathcal{C} \to \mathcal{C}$, an isomorphism $\nu: \mathbb{1} \to \mathbb{1}^*$, morphisms $\{b_V: \mathbb{1} \to V \otimes V^*, \tau_V: V \to V^{**}\}_{V\in\mathcal{C}}$, and invertible morphisms $\{\gamma_{V,W}: V^* \otimes W^* \to (W \otimes V)^*\}_{V,W\in\mathcal{C}}$ satisfying axioms (1) and (2) above. Barrett and Westbury do not suppose the existence of morphisms d forming together with b a duality in $\mathcal{C}$. However, this can be deduced from their axioms. We assume here from the beginning that $\mathcal{C}$ has a left duality and extract from it the duality functor $*$ and the isomorphisms γ and ν. Our axioms imply the Barrett–Westbury axioms. Only condition (4) in [BW3], p. 362, is somewhat involved; its proof uses the morphisms $\tilde{b}$, $\tilde{d}$ defined above.

3 Invariants of 3-dimensional G-manifolds from spherical G-categories. The state sum approach to topological invariants of 3-dimensional manifolds was introduced in [TV] and extended in [BW2] (see also [GK]). Barrett and Westbury [BW2] showed that any finite semisimple spherical category $\mathcal{C}$ over K such that $\mathrm{End}(\mathbb{1}_{\mathcal{C}}) = K$ gives rise to a K-valued invariant of closed oriented 3-dimensional manifolds. We explain how to adapt the methods of [TV], [BW2] in order to derive from a finite semisimple spherical G-category $\mathcal{C}$ a topological invariant $|M|_{\mathcal{C}} \in K$ of a closed oriented 3-dimensional G-manifold M. Here a G-category $\mathcal{C}$ is *finite semisimple* if it satisfies Axioms (1.1.1)–(1.1.4), p. 158, $\mathcal{C}_\alpha \neq \emptyset$ for all $\alpha \in G$, and the sum (1.7.a) from Chapter VII yields an invertible element D^2 of K. For $\alpha \in G$, denote by I_α the set of isomorphism classes of simple objects of $\mathcal{C}_\alpha$. The same argument as in Section VII.1.7 shows that the sum (1.7.b) (Chapter VII) is equal to D^2 for all $\alpha \in G$.

Fix a triangulation T of M. Pick a map $g: M \to K(G,1)$ representing the given homotopy class of maps and carrying all vertices of T to the base point of $K(G,1)$.

We assign to each edge e of T the element $g_e \in G$ represented by the loop $g|_e$. In terminology of Section IV.3.1, the function $e \mapsto g_e$ is a G-system on T representing g. We define $|T|_{\mathcal{C}} \in K$ as in [BW2] with the difference that we allow only the labelings assigning to each edge e of T an element of $I_{g(e)}$. As in [BW2], we use a *local order* on T, that is, a choice of a total order on the set of vertices of each simplex Δ of T such that the order on the vertices of any subsimplex of any Δ is induced by the order on the vertices of Δ. For instance, a total order on the set of all vertices of T induces a local order on T.

We claim that $|M|_{\mathcal{C}} = |T|_{\mathcal{C}}$ is a well-defined topological invariant of M, i.e., it does not depend on the choice of T, the choice of the local order, and the choice of g in its homotopy class. Note first that any Pachner move on T extends in a natural way to the G-systems on T. This extension preserves the values of the G-systems on all edges kept under the move. Now, the arguments of [BW2] show that $|T|_{\mathcal{C}}$ is invariant under the Pachner moves and does not depend on the local order on T. Different choices of g lead to G-systems on T homotopic in the sense of Section IV.3.1, where $T_\bullet = \emptyset$. The rest of the proof goes by reformulating $|T|_{\mathcal{C}}$ in terms of skeletons of M. A *skeleton* of M is a finite 2-polyhedron in M whose complement is a disjoint union of open 3-balls. For instance, the closed 2-cells in M dual to the edges of T form a skeleton $T^* \subset M$. Using this observation, we can dualize the notion of a G-system from triangulations to skeletons of M. (A G-system on a skeleton assigns elements of G to oriented 2-faces). If T is locally ordered, then the order determines an orientation of every edge e of T. Since M is oriented, the orientation of e induces an orientation on the 2-cell dual to e. This turns T^* into an oriented branched 2-polyhedron in the sense of [BPe]. We can switch from the language of state sums on triangulations to the one of state sums on oriented branched skeletons endowed with G-systems. In the latter language, a homotopy move on G-systems can be decomposed into a composition of more elementary Matveev–Piergallini moves and bubblings, cf. [TV], [BPe], and Chapter IV. The invariance of the state sum under these moves is verified as in [BW2].

The invariant $|M|_{\mathcal{C}}$ is conjecturally related to the invariants of M derived from modular G-categories in Chapter VII: any finite semisimple spherical G-category $\mathcal{C}$ presumably gives rise to a modular G-category $\mathcal{Z}$ (the "center" or the "double" of $\mathcal{C}$) such that $|M|_{\mathcal{C}} = \tau_{\mathcal{Z}}(M)$ for all M.

4 Spherical Hopf algebras. The notion of a spherical category has a counterpart in the theory of Hopf algebras. Following [BW3] we call a *spherical Hopf algebra* a Hopf algebra A over K endowed with a group-like element $w \in A$ such that the square of the antipode in A is equal to the conjugation by w and $\mathrm{Tr}(fw) = \mathrm{Tr}(fw^{-1})$ for any A-linear endomorphism f of an A-module whose underlying K-module is projective of finite type. We can extend this definition to Hopf G-coalgebras. A *spherical Hopf G-coalgebra* is a Hopf G-coalgebra $(\{A_\alpha\}_{\alpha\in G}, \Delta, \varepsilon_1, s, \varphi)$ endowed with invertible elements $\{w_\alpha \in A_\alpha\}_{\alpha\in G}$ such that for any $\alpha, \beta \in G$, $a \in A_\alpha$,

$$S_{\alpha^{-1}} S_\alpha(a) = w_\alpha a w_\alpha^{-1},\ \Delta_{\alpha,\beta}(w_{\alpha\beta}) = w_\alpha \otimes w_\beta,\ \varepsilon_1(w_1) = 1,\ \varphi_\alpha(w_\beta) = w_{\alpha\beta\alpha^{-1}},$$

and $\mathrm{Tr}(f w_\alpha) = \mathrm{Tr}(f w_\alpha^{-1})$ for any A_α-linear endomorphism f of an A_α-module whose underlying K-module is projective of finite type. Note the useful identity $S_\alpha(w_\alpha) = w_{\alpha^{-1}}^{-1}$.

It is clear that an involutory Hopf G-coalgebra (so that $S_{\alpha^{-1}} S_\alpha = \mathrm{id}$ for all $\alpha \in G$) is spherical with $w_\alpha = 1_\alpha \in A_\alpha$ for all $\alpha \in G$. A ribbon Hopf G-coalgebra (A, θ) is spherical with $w_\alpha = \theta_\alpha u_\alpha$ where $u_\alpha \in A_\alpha$ is the (generalized) Drinfeld element of R (see Appendix 7).

It is proven in [BW3] that for a spherical Hopf algebra (A, w), the monoidal category $\mathrm{Rep}(A)$ is spherical. For $V \in \mathrm{Rep}(A)$, the morphism $\tau_V : V \to V^{**}$ is defined as the standard identification $V = V^{**}$ followed by multiplication by w^{-1}. Similarly, for a spherical Hopf G-coalgebra (A, w), the category $\mathrm{Rep}(A)$ is spherical: for $V \in \mathrm{Rep}(A_\alpha)$, the morphism $\tau_V : V \to V^{**}$ is defined as the standard identification $V = V^{**}$ followed by multiplication by w_α^{-1}.

Appendix 3

Recent work on HQFTs

We outline the work on HQFTs and on related algebraic structures done since the appearance of my first preprints on this subject [Tu3], [Tu4].

In the 2-dimensional case, the notion of an HQFT was introduced independently of [Tu3] by M. Brightwell and P. Turner [BT1]. These authors classified 2-dimensional HQFTs with simply-connected targets in terms of Frobenius algebras. The role of 2-categories in this setting was discussed in their subsequent paper [BT2]. Two different approaches to 2-dimensional HQFTs with arbitrary targets were introduced in [PT] and [ST]; see also [Po]. Relations between 2-dimensional HQFTs and gerbes were discussed in [BTW].

Relative 2-dimensional HQFTs with target $(X = K(G,1), x, X, x)$ were introduced and studied by G. Segal and G. Moore; see [Mo], [MS]. These authors also gave a new, geometric proof of Theorem II.6.3 and Corollary III.3.2 first established in [Tu3]. Crossed Frobenius G-algebras were generalized to so-called G-twisted Frobenius algebras in [Kau].

The axiomatic definition of HQFTs, introduced in [Tu3], was investigated and improved by G. Rodrigues [Rod]. A related notion of a homological quantum field theory was introduced by E. Castillo and R. Diaz [CD].

M. Müger [Mü4] showed that Galois extensions of (usual) braided tensor categories have a natural structure of braided crossed G-categories. In [Mü5], Müger established a fundamental connection between 1-dimensional quantum field theories and braided crossed G-categories. Namely, a quantum field theory on the real line having a group G of inner symmetries gives rise to a braided crossed G-category (the category of twisted representations). Its neutral subcategory is equivalent to the usual representation category of the theory. Combining with the results of this book we obtain the following chain of constructions:

a Rational Conformal Field Theory with group of inner symmetries G
$\downarrow$
a crossed modular G-category
$\downarrow$
a 3-dimensional HQFT with target $K(G,1)$

A brief exposition of results of Müger is given in Appendix 5.

A. Virelizier [Vir1]–[Vir4] studied algebraic properties and topological applications of crossed Hopf G-coalgebras. In [Vir2], Virelizier showed the existence of integrals and traces for such coalgebras and generalized to them the main properties of usual quasitriangular and ribbon Hopf algebras. In [Vir3], he used involutary Hopf G-coalgebras to define a scalar invariant of flat G-bundles over 3-manifolds, generalizing a construction due to G. Kuperberg. A new method producing non-trivial examples of quasitriangular crossed Hopf G-coalgebras was given in [Vir4]. An exposition of the work of Virelizier is given in Appendices 6 and 7.

M. Zunino [Zu1] generalized the Drinfeld quantum double of Hopf algebras to crossed Hopf G-coalgebras. Zunino's double $D(H)$ of a crossed Hopf G-coalgebra H is a quasitriangular crossed Hopf G-coalgebra. He showed that if G is finite and H is semisimple, then $D(H)$ is modular. In [Zu2], Zunino defined a double for crossed G-categories and established its compatibility with representation theory: for a crossed Hopf G-coalgebra H, the representation category of $D(H)$ is equivalent to the double of the representation category of H. Symbolically $\operatorname{Rep} D(H) \approx D(\operatorname{Rep} H)$. This shows that Zunino's double keeps the main features of the Drinfeld double.

A. Kirillov, Jr., [Kir2] pointed out that for a vertex operator algebra V satisfying suitable assumptions and a finite group G of automorphisms of V, the category of twisted V-modules is a braided crossed G-category. Kirillov showed that every braided crossed G-category has a natural quotient, called the orbifold category, that is, a braided monoidal category in the usual sense of the word. Kirillov also introduced and studied an interesting extension of our algebra of colors, which he called extended Verlinde algebra.

Braided crossed G-categories were further investigated in [DGNO]. A method producing ribbon crossed G-categories from a simple complex Lie algebra $\mathfrak{g}$ with center G was suggested in [LT]. It is based on a study of representations of the quantum group $U_q(\mathfrak{g})$ at roots of unity. Analogues of the Yetter–Drinfeld modules in the context of braided crossed G-categories were studied by F. Panaite and M. Staic [PS].

Hopf G-coalgebras were further studied by A. Hegazi and co-authors [HMM], [HIE] and by S.-H. Wang [Wa1]–[Wa4], see also [LC]. A categorical approach to Hopf G-coalgebras was introduced by S. Caenepeel and M. De Lombaerde [CL].

A generalization of Hopf G-coalgebras to so-called G-cograded multiplier Hopf algebras was introduced and studied by A. Van Daele, L. Delvaux and their co-authors [ADV], [De], [DV], [DVW], [VW1], [VW2]; see also [HE], [LW], [SW].

For a related work on topological invariants of G-tangles derived from factorized groups see [KR]. For an explicit construction of HQFTs based on the Dijkgraaf–Witten technique see [HST].

Appendix 4
Open problems

1. *Classification of crossed G-algebras.* Classify crossed G-algebras for sufficiently simple groups G, say, for cyclic, abelian, finite groups. Find interesting examples of non-semisimple crossed G-algebras.

2. *Deformations.* Deformations of Frobenius algebras form a subject of a deep theory based on the Witten-Dijkgraaf-Verlinde-Verlinde equation; see, for instance, [Du]. It would be interesting to generalize this theory to crossed G-algebras.

3. *Modular G-categories.* Produce modular G-categories giving non-trivial topological invariants that allow to distinguish 3-dimensional G-manifolds.

4. *Relations between various approaches.* In the theory of quantum invariants of 3-dimensional manifolds, there is a fundamental relationship between the surgery approach and the state sum approach; see [Tu2], [Ro]. It would be interesting to generalize this to HQFTs. A conjecture in this direction is stated in Appendix 2, see also Appendix 7.

5. *Invariants of spin-structures.* Instead of maps from a manifold to a fixed target space one can consider maps whose source is the oriented frame bundle of the manifold (the principal bundle of positively oriented bases in the tangent spaces of points). When the target space is $K(G,1)$, this should lead to new algebraic notions generalizing G-algebras and crossed G-categories. The resulting theory should include the quantum invariants of spin-structures on closed oriented 3-manifolds introduced in [Bl], [KM], [Tu1].

6. *Subfactors and planar algebras.* A. Ocneanu showed that the subfactors of type II_1 give rise to topological 3-manifold invariants via the state sum approach; see, for instance, [EK]. What is the counterpart of the equivariant theory introduced here in the setting of subfactors? Similarly, the planar algebras introduced by V. Jones [Jon] are intimately related to TQFTs. What is the counterpart of HQFTs for planar algebras?

7. *Perturbative aspects.* What are the perturbative aspects of HQFTs? Are there perturbative invariants of 3-dimensional G-manifolds generalizing the Le–Murakami–Ohtsuki invariant [LMO]?

8. *Higher-dimensional generalizations.* Quantum invariants of 3-manifolds have 4-dimensional counterparts; see [CKY], [CKS], [CJKLS], [Ma], and references therein. The role of categories in these constructions is played by 2-categories. The homotopy

quantum field theory should have similar 4-dimensional and high-dimensional versions and give rise to a notion of a crossed 2-category.

In this book we consider mainly target spaces of type $K(G, 1)$. In dimension four it may be more appropriate to consider target spaces of type $K(H, 2)$, where H is an abelian group. Note that 2-dimensional HQFTs with such target space were studied in [BT1].

It is interesting to describe algebraic data yielding invariants of $Spin^c$-structures on 4-manifolds. As the sources of maps one can take the oriented frame bundles of 4-manifolds and as the target space $K(H, 2)$. Can one include the Seiberg–Witten invariants in this framework?

9. *Relative HQFTs.* Verify that the functor from the category of 2-dimensional relative HQFTs to the category of Frobenius triples defined in Section 3 of Appendix 1 is an equivalence of categories.

10. *Cobordism Hypothesis.* Generalize the Baez–Dolan Cobordism Hypothesis for extended TQFTs (see [BD], [Lu]) to HQFTs.

11. *Miscellaneous.* Study the representations of the mapping class groups resulting from HQFTs, cf. [Kir2] in the case of the torus. Find an interpretation of HQFTs in terms of algebraic geometry, i.e., in terms of sections of bundles over moduli spaces. Study the 2-dimensional homotopy modular functors and their relations to quantum computations, cf. [FKW]. Study relations with number theory, cf. [LZ]. Is there a homotopy version of the Khovanov link homology?

Appendix 5
On the structure of braided crossed G-categories

by Michael Müger

The main aim of this appendix is to discuss, for any finite group G, a close connection between braided crossed G-categories and ribbon categories containing the representation category of G as a full braided subcategory. In fact, in the context of finite semisimple categories, this will turn out to be a bijection (modulo suitable equivalences). As an application we prove that every braided G-crossed fusion category is equivalent to a strict monoidal category with strict G-action, justifying the strictness assumption made in Chapters VI and VII of this book. In the last section we touch upon the open problem of obtaining braided crossed G-categories as 'crossed products' of braided categories by finite group actions. The existence of such crossed products is shown to be equivalent to the conjectured existence of embeddings of braided fusion categories into modular categories of minimal size.

The original references on which this appendix is based are [Bru] by Bruguières, [Kir1], [Kir2] by Kirillov, Jr., and [Mü1], [Mü4], [Mü5] by the author. For a more extensive treatment of some of the matters discussed in this appendix, cf. [DGNO], in particular Section 4, "Equivariantization and de-equivariantization".

5.1 Braided crossed G-categories

1.1 Definition. Let $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ be a monoidal category (with associativity constraint a and left and right unit constraints l, r). An automorphism of $\mathcal{C}$ is a monoidal functor (β, γ, σ) from $\mathcal{C}$ to itself, where β is a self-equivalence of $\mathcal{C}$, γ is a natural family $\{\gamma_{X,Y} : \beta(X \otimes Y) \to \beta(X) \otimes \beta(Y)\}$ of isomorphisms and $\sigma : \beta(\mathbb{1}) \to \mathbb{1}$ an isomorphism. For the exact conditions on β, γ, σ, cf. [Mac]. The composition $(\beta, \gamma, \sigma) \circ (\beta', \gamma', \sigma')$ is defined as $(\beta \circ \beta', \gamma'', \sigma'')$ with $\sigma'' = \sigma \circ \beta(\sigma')$ and γ'' defined by the composition

$$\gamma''_{X,Y} : \beta(\beta'(X \otimes Y)) \xrightarrow{\beta(\gamma'_{X,Y})} \beta(\beta'(X) \otimes \beta'(Y)) \xrightarrow{\gamma_{\beta'(X),\beta'(Y)}} \beta(\beta'(X)) \otimes \beta(\beta'(Y)).$$

When $\mathcal{C}$ is braided, automorphisms of $\mathcal{C}$ are also required to respect the braiding, i.e., satisfy $\beta(c_{X,Y}) = c_{\beta(X),\beta(Y)}$. The set of (braided) automorphisms of $\mathcal{C}$ is denoted by $\operatorname{Aut} \mathcal{C}$.

Now $\operatorname{Aut} \mathcal{C}$ is the categorical group (i.e., monoidal category where every morphism is invertible and for every object X there is an object Y such that $X \otimes Y \cong \mathbb{1}$) having automorphisms of $\mathcal{C}$ as objects and natural monoidal isomorphisms (i.e., monoidal natural transformations all components of which are isomorphisms) of monoidal functors as morphisms.

The most concise way of defining an action of a (discrete) group on a monoidal category is as a monoidal functor:

1.2 Definition. If G is a group, let $\mathcal{G}$ be the discrete category (the only morphisms are the identity morphisms) with $\mathrm{Obj}\,\mathcal{G} = G$ and the obvious strict tensor product. An action β of G on a (braided) tensor category $\mathcal{C}$ is a monoidal functor $\beta\colon \mathcal{G} \to \mathrm{Aut}\,\mathcal{C}$, $g \mapsto \beta_g$. We usually abbreviate by writing ${}^gX = \beta_g(X)$, etc.

Since each β_g is a monoidal functor, it comes with natural isomorphisms

$$\gamma_{g,X,Y} : \beta_g(X \otimes Y) \to \beta_g(X) \otimes \beta_g(Y) \quad \text{and} \quad \sigma_g : \beta_g(\mathbb{1}) \to \mathbb{1}.$$

On the other hand, the monoidality of the functor $\beta\colon \mathcal{G} \to \mathrm{Aut}\,\mathcal{C}$ provides natural monoidal isomorphisms $\delta_{g,h}\colon \beta_{gh} \to \beta_g \circ \beta_h$ and $\varepsilon\colon \beta_e \to \mathrm{id}_{\mathcal{C}}$, or, in terms of components, $\delta_{g,h,X} : {}^{(gh)}X \to {}^g({}^hX)$ and $\varepsilon_X : {}^eX \to X$. One can easily unpack the definition to obtain the identities satisfied by these isomorphisms.

1.3 Definition. A G-action β on a strict monoidal category is called strict if all isomorphisms $\gamma_{g,X,Y}$, $\delta_{g,h,X}$, σ_g, ε_X are identities.

1.4 Definition. If $\mathcal{C}$ is a monoidal category and G a group, a G-grading on $\mathcal{C}$ is a map $\partial\colon \mathrm{Obj}\,\mathcal{C} \to G$ such that $\partial(X \otimes Y) = \partial X \cdot \partial Y$ and $\partial X = \partial Y$ whenever $X \cong Y$. The image of ∂ is called the G-spectrum of the G-graded monoidal category $\mathcal{C}$. The grading is called trivial or full if the G-spectrum equals $\{e\}$ or G, respectively.

1.5 Remark. 1. In the categorical literature, also G-gradings on the morphisms are considered. We, however, consider only gradings on the objects.

2. One could try to define a G grading to be a monoidal functor $\partial\colon \mathcal{C} \to \mathcal{G}$, but in the k-linear case, where $\mathrm{Hom}(X, Y)$ is never empty, this does not work since it would imply that all objects have degree e.

3. The above definition rules out direct sums of objects of different degrees. Often, however, it is desirable to work with semisimple categories, which includes having all direct sums. This can be accommodated by only requiring the existence of a full monoidal subcategory $\mathcal{C}_{\mathrm{hom}} \subset \mathcal{C}$ of *homogeneous* objects such that (a) $\mathcal{C}_{\mathrm{hom}}$ satisfies Definition 1.4, and (b) every object in $\mathcal{C}$ is a finite direct sum of objects in $\mathcal{C}_{\mathrm{hom}}$.

The following two definitions first appeared in [Tu4], underlying Chapter VI of this book; cf. also [CM].

1.6 Definition. A crossed G-category is a monoidal category together with a G-action β and a G-grading ∂ such that $\partial({}^gX) = g\,\partial X g^{-1}$. We define full subcategories by $\mathcal{C}_g = \partial^{-1}(g)$ and notice that the G-action leaves $\mathcal{C}_e$ stable.

1.7 Remark. The spectrum of a rigid G-graded monoidal category is a subgroup of G. In the case of a crossed G category it is a normal subgroup.

1.8 Definition. A braiding on a crossed G-category $(\mathcal{C}, \beta, \partial)$ is a family of natural isomorphisms $\{c_{X,Y} : X \otimes Y \to {}^{\partial X}Y \otimes X\}_{X,Y \in \mathcal{C}_{\text{hom}}}$ satisfying naturality in the sense that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & {}^{\partial X}Y \otimes X \\ {\scriptstyle s\otimes t}\downarrow & & \downarrow{\scriptstyle {}^{\partial X}t \otimes s} \\ X' \otimes Y' & \xrightarrow{c_{X',Y'}} & {}^{\partial X'}Y' \otimes X' \end{array}$$

commutes for all $s \in \text{Hom}(X, X')$, $t \in \text{Hom}(Y, Y')$, as well as commutativity of

$$\begin{array}{ccccccc} (X \otimes Y) \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & {}^{gh}Z \otimes (X \otimes Y) & \xrightarrow{a^{-1}} & ({}^{gh}Z \otimes X) \otimes Y & & \\ {\scriptstyle a}\downarrow & & & & & \searrow{\scriptstyle (\delta_{g,h,Z} \otimes \text{id}) \otimes \text{id}} & \\ X \otimes (Y \otimes Z) & \xrightarrow{\text{id} \otimes c_{Y,Z}} & X \otimes ({}^{h}Z \otimes Y) & \xrightarrow{a^{-1}} & (X \otimes {}^{h}Z) \otimes Y & \xrightarrow{c_{X,{}^{h}Z} \otimes \text{id}} & ({}^{g}({}^{h}Z) \otimes X) \otimes Y \end{array}$$

for all $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $Z \in \mathcal{C}_k$ and of a similar diagram involving $c_{X,Y\otimes Z}$.

1.9 Remark. Imposing naturality, one can uniquely extend $c_{X,Y}$ to the situation where $X \in \mathcal{C}_{\text{hom}}$, $Y \in \mathcal{C}$, but the requirement $X \in \mathcal{C}_{\text{hom}}$ cannot be relaxed.

5.2 The G-fixed category of a braided crossed G-category

The following construction is well known, but it is hard to find the first reference.

2.1 Definition/Proposition. *Let $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ be a monoidal category. Let β be an action of the group G on $\mathcal{C}$. Then $\mathcal{C}^G$ is the monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)^G = (\mathcal{C}^G, \otimes^G, \mathbb{1}^G, a^G, l^G, r^G)$ defined as follows: Its objects are pairs $(X, \{u_g\}_{g\in G})$, where $X \in \mathcal{C}_{\text{hom}}$ and, for each $g \in G$, $u_g : {}^{g}X \to X$ is an isomorphism such that the diagram*

$$\begin{array}{ccc} {}^{gh}X & \xrightarrow{\delta_{g,h,X}} & {}^{g}({}^{h}X) \\ {\scriptstyle u_{gh}}\downarrow & & \downarrow{\scriptstyle {}^{g}(u_h)} \\ X & \xleftarrow[u_g]{} & {}^{g}X \end{array} \tag{2.1}$$

commutes for all $g, h \in G$. The Hom-sets are defined by

$$\text{Hom}_{\mathcal{C}^G}((X, u), (Y, v))$$
$$= \left\{ s \in \text{Hom}_{\mathcal{C}}(X, Y) \;\middle|\; \textit{the diagram} \begin{array}{ccc} {}^{g}X & \xrightarrow{{}^{g}s} & {}^{g}Y \\ {\scriptstyle u_g}\downarrow & & \downarrow{\scriptstyle v_g} \\ X & \xrightarrow[s]{} & Y \end{array} \textit{commutes for all } g \in G \right\}.$$

The tensor product of objects is defined by $(X,u)\otimes(Y,v)=(X\otimes Y,w)$*, where* $w_\bullet$ *is given by*

$$w_g : {}^g(X\otimes Y) \xrightarrow{\gamma_{g,X,Y}} {}^gX\otimes{}^gY \xrightarrow{u_g\otimes v_g} X\otimes Y.$$

The tensor product of morphisms is inherited from $\mathcal{C}$*. The monoidal unit* $\mathbb{1}^G$ *is given by* $(\mathbb{1},\{\sigma_g\})$*, where* $\sigma_g : {}^g\mathbb{1}\to\mathbb{1}$ *is the isomorphism* ${}^g\mathbb{1}\to\mathbb{1}$ *coming with the monoidal functor* $\beta\colon \mathcal{G}\to \operatorname{Aut}\mathcal{C}$*. The associativity constraint* a^G *is given by*

$$a^G((X,w^X),(Y,w^Y),(Z,w^Z)) = a(X,Y,Z),$$

and similarly for the unit constraints l^G, r^G*.*

2.2 Remark. The correct name for $\mathcal{C}^G$ would be 'category of G-modules in $\mathcal{C}$', but it seems to be more customary to speak of the 'G-fixed' category. In [DGNO], the passage $\mathcal{C}\rightsquigarrow\mathcal{C}^G$ is called 'equivariantization'.

2.3 Proposition. *Let* $(\mathcal{C},\beta,\partial,c)$ *be a braided crossed* G*-category. Then* $\mathcal{C}^G$ *is braided with braiding* c^G *given by*

$$c^G_{(X,u),(Y,v)} : X\otimes Y \xrightarrow{c_{X,Y}} {}^{\partial X}Y\otimes X \xrightarrow{v_{\partial X}\otimes \mathrm{id}_X} Y\otimes X.$$

Proof. One must show that c_G is natural w.r.t. both variables and satisfies both braid (or hexagon) equations. This amounts to straightforward combinations of the properties of c and the definition of $\mathcal{C}^G$ and of c^G. We omit the details. □

The notation ${}^gX$ for $\beta(g)(X)$ used above is convenient, but it hides the dependence on the choice of a functor $\beta\colon\mathcal{G}\to\operatorname{Aut}\mathcal{C}$. In principle, we should write $\mathcal{C}^{(G,\beta)}$ instead of $\mathcal{C}^G$. We will do so only in the formulation of the following result.

2.4 Lemma. *Let* $\beta_1,\beta_2\colon\mathcal{G}\to\operatorname{Aut}\mathcal{C}$ *be actions of a group* G *on a monoidal category* $\mathcal{C}$*. A natural monoidal isomorphism* $\beta_1\cong\beta_2$ *of monoidal functors induces a monoidal equivalence* $\mathcal{C}^{(G,\beta_1)}\simeq\mathcal{C}^{(G,\beta_2)}$ *between the respective fixpoint categories.*

Defining functors of categories carrying a G-action is not entirely trivial, but for our purposes it is sufficient to have a notion of equivalence of monoidal G-categories. To this purpose, we observe that given a monoidal equivalence $F\colon\mathcal{C}\to\mathcal{D}$, there is an adjoint equivalence $G\colon\mathcal{D}\to\mathcal{C}$, unique up to natural monoidal isomorphism. Therefore, if $E\in\operatorname{Aut}\mathcal{C}$ then $F\circ E\circ G\in\operatorname{Aut}\mathcal{D}$, and this gives rise to a monoidal equivalence $\widetilde{F}\colon \operatorname{Aut}\mathcal{C}\to\operatorname{Aut}\mathcal{D}$.

2.5 Definition. Let $\mathcal{C},\mathcal{D}$ be monoidal categories carrying G-actions β,β'. Then an equivalence of $\mathcal{C},\mathcal{D}$ (as monoidal G-categories) is a monoidal equivalence $F\colon\mathcal{C}\to\mathcal{D}$ such that the monoidal functors $\widetilde{F}\circ\beta$ and β' (both from $\mathcal{G}$ to $\operatorname{Aut}\mathcal{D}$) are monoidally equivalent.

A functor of G-graded monoidal categories is a monoidal functor $F\colon\mathcal{C}\to\mathcal{D}$ such that $F(\mathcal{C}_{\mathrm{hom}})\subset\mathcal{D}_{\mathrm{hom}}$ and $\partial_{\mathcal{D}}F(X)=\partial_{\mathcal{C}}X$ for all $X\in\mathcal{C}_h$.

Combining Lemma 2.4 with Definition 2.5, one finds:

2.6 Proposition. *An equivalence $E: \mathcal{C} \to \mathcal{D}$ of braided crossed G-categories gives rise to an equivalence $E^G: \mathcal{C}^G \to \mathcal{D}^G$ of braided categories.*

Up to this point, our considerations were completely general in that we made no further assumptions on the categories or the groups. From now on we will restrict ourselves to finite groups and semisimple k-linear categories over an algebraically closed field k of characteristic zero.

2.7 Proposition. *Let G be a finite group, k an algebraically closed field of characteristic zero and $\mathcal{C}$ a semisimple k-linear monoidal category carrying a G-action. Then $\mathcal{C}^G$ is a semisimple monoidal category having a full monoidal subcategory $\mathcal{S} \simeq \mathrm{Rep}_k\, G$. If $\mathcal{C}$ is braided or, more generally braided G-crossed, then $\mathcal{S}$ is a braided subcategory of $\mathcal{C}^G$. Cf.* [Kir1], [DGNO].

We see that a braided G-crossed category gives rise to a braided category $\mathcal{C}^G$ containing $\mathrm{Rep}_k\, G$ as full subcategory. In the next section, we will consider a construction that goes the opposite way. We will limit ourselves to the setting of the following definition:

2.8 Definition. Let k be an algebraically closed field of characteristic zero. A fusion category over k is a k-linear semisimple ribbon braided tensor category with simple unit, i.e., $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = k\mathrm{id}_{\mathbb{1}}$, and finitely many isomorphism classes of simple objects.

2.9 Remark. 1. One may also consider non-braided fusion categories, in which case the definition of rigidity requires attention, one approach being the spherical categories of [BW1]; cf. Appendix 2.

2. In fact, braided spherical categories are the same as ribbon categories.

5.3 From braided categories containing Rep G to braided G-crossed categories

The following definition is a straightforward generalization of notions from ordinary algebra:

3.1 Definition. Let $\mathcal{C}$ be a strict monoidal category. An algebra (= monoid) in $\mathcal{C}$ is a triple (A, m, η), where A is an object and $m: A \otimes A \to A$, $\eta: \mathbb{1} \to A$ are morphisms satisfying $m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)$ and $m \circ (\eta \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$. (In the non-strict case, one has to insert the associativity constraint at the obvious place.) If $\mathcal{C}$ has a braiding c, then an algebra in $\mathcal{C}$ is called commutative if $m \circ c_{A,A} = m$. A commutative algebra is called étale if there is a morphism $\Delta: A \to A \otimes A$ that satisfies $m \circ \Delta = \mathrm{id}_A$ and is a morphism of A-A bimodules, i.e.

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = m \circ \Delta = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta).$$

The study of (commutative) algebras in monoidal categories, e.g. those associated to quantum groups, is a very interesting subject and was used to great effect in [Del]. However, we will only need the following example arising from representation categories of finite groups:

3.2 Proposition. *Let G be a finite group and k an algebraically closed field of characteristic zero. The symmetric monoidal category $\mathcal{S} = \mathrm{Rep}_k\, G$ of finite dimensional representations of G on k-vector spaces contains a commutative algebra (A, m, η) with the following properties:*

(i) $\dim_k A = |G|$.

(ii) $\dim \mathrm{Hom}_{\mathcal{S}}(\mathbb{1}, A) = 1$.

(iii) *The object A is 'absorbing': $A \otimes X \cong A^{\oplus \dim X}$ for all $X \in \mathcal{S}$.*

(iv) *There is an isomorphism $G \xrightarrow{\cong} \mathrm{Aut}(A, m, \eta) \equiv \{s \in \mathrm{Aut}\, A \mid m \circ (s \otimes s) = s \circ m,\ s \circ \eta = \eta\}$.*

(v) *The algebra (A, m, η) is étale.*

Proof. Let $A = \mathrm{Fun}(G, k)$ with algebra structure given by pointwise multiplication with the constant function 1 as unit. With the G-action $(\pi_l(g) f)(h) = f(g^{-1}h)$, this is the left regular representation π_l of G, which is well known to have properties (i) through (iii). ((i) is obvious, (ii) holds since the subspace of G-stable elements of A is one-dimensional (the constant functions), and (iii) follows from the fact that $A \cong \bigoplus_i \dim(X_i) \cdot X_i$, where X_i runs through the irreducible representations X_i.) For claim (iv), cf. e.g. [Mü4], Remark 2.9. With $\Delta(f)(g, h) = \delta_{g,h} f(g)$ for $f \in A = \mathrm{Fun}(G, k)$, the last statement holds by easy computations. □

3.3 Remark. For most applications in this appendix, Proposition 3.2 will be sufficient. In several other, but closely related, applications we are confronted by symmetric monoidal categories that are not a priori known to be of the form Rep G. It is therefore important that every k-linear rigid symmetric monoidal category with simple unit, finitely many simple objects and trivial twists is equivalent to Rep G for a finite group G that is unique up to isomorphisms. (For stronger results without finiteness assumption cf. [Del] and, in the case of $*$-categories, [DR1]. An exposition of the result for $*$-categories can be found in [Mü6].) As to the last requirement, recall that every object in a symmetric ribbon category comes with a twist automorphism $\theta(X)$ of order two. In particular, for a simple object X we have $\theta(X) = \pm \mathrm{id}_X$, and the category is called even if all twists are identities. All these results have suitable generalizations to the non-even case.

Again, as in commutative algebra, one defines

3.4 Definition. A (left) module over an algebra (A, m, η) is a pair (X, μ) where $X \in \mathcal{C}$ and $\mu\colon A \otimes X \to X$ satisfies $\mu \circ (\mathrm{id}_A \otimes \mu) = \mu \circ (m \otimes \mathrm{id}_X)$. The left modules form a category ${}_A\mathcal{C} = A\text{-}\mathrm{Mod}_{\mathcal{C}}$ with Hom-sets defined by

$$\mathrm{Hom}_{A\text{-}\mathrm{Mod}_{\mathcal{C}}}((X, \mu), (X', \mu')) = \{s \in \mathrm{Hom}(X, X') \mid s \circ \mu = \mu' \circ (\mathrm{id}_A \otimes s)\}.$$

Under a very weak condition on a braided category $\mathcal{C}$ (existence of coequalizers), which is satisfied in any abelian category, one finds that the category of modules over a commutative algebra in $\mathcal{C}$ is monoidal, the definition of the tensor product being a natural generalization of the usual one. The monoidal unit of ${}_A\mathcal{C}$ is (A, m), and there is a canonical monoidal functor $F : \mathcal{C} \to {}_A\mathcal{C}$ such that $X \mapsto (A \otimes X, m \otimes \mathrm{id}_X)$. Assuming $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = k$, one finds that $\mathrm{End}_{{}_A\mathcal{C}}(A) = k$ holds if and only if $\dim \mathrm{Hom}(\mathbb{1}, A) = 1$.

In order that ${}_A\mathcal{C}$ be semisimple, some technical condition on the algebra A is needed. One suitable notion is the étaleness defined above, the terminology being motivated by the corresponding notion in commutative algebra, cf. [DMNO]. Similar, but slightly differently formulated conditions were considered in [Bru], [KO], [Mü2]. In the context of fusion categories, these conditions are equivalent, and they imply in particular that the functor F is *dominant*, i.e., every simple object of ${}_A\mathcal{C}$ is a direct summand of $F(X)$ for some $X \in \mathcal{C}$.

However, the braiding of $\mathcal{C}$ does not unconditionally descend to a braiding on ${}_A\mathcal{C}$.

3.5 Definition ([Mü1])**.** The symmetric center $\mathcal{Z}_2(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ is the full subcategory consisting of those objects X that satisfy

$$c(X, Y) \circ c(Y, X) = \mathrm{id}_{Y \otimes X} \quad \text{for all } Y \in \mathcal{C}.$$

3.6 Remark. 1. The symmetric center is a symmetric monoidal category. It coincides with $\mathcal{C}$ if and only if $\mathcal{C}$ is symmetric.

2. One can show that a braided fusion category is modular if and only if its symmetric center is trivial, in that it contains only direct multiples of the unit object. (The 'only if' direction is immediate; for the 'if', see [BeBl]; cf. also [Re] for a similar result in the context of endomorphisms of a C^*-algebra.)

As shown in [Mü1], the obvious candidate for a braiding ${}_A\mathcal{C}$ actually is a braiding if and only if $A \in \mathcal{Z}_2(\mathcal{C})$. If this is not the case, naturality of the putative braiding w.r.t. one of the arguments fails.

Let now $\mathcal{C}$ be a braided monoidal category with a full braided monoidal subcategory $\mathcal{S} \simeq \mathrm{Rep}_k\, G$. Now Proposition 3.2 gives rise to a commutative étale algebra (A, m, η) in $\mathcal{S}$ and thus in $\mathcal{C}$. Since A has every simple object of $\mathcal{S}$ as a direct summand, we have $A \in \mathcal{Z}_2(\mathcal{C})$ if and only if $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$. Under this assumption, ${}_A\mathcal{C}$ is braided. The functor $F : \mathcal{C} \to {}_A\mathcal{C}$ has the property that it trivializes $\mathcal{S}$, in the sense that $F(X)$ is a direct multiple of $\mathbb{1}$ for every $X \in \mathcal{S}$. What is more, when $\mathcal{S} = \mathcal{Z}_2(\mathcal{C})$ then ${}_A\mathcal{C}$ has trivial center $\mathcal{Z}_2({}_A\mathcal{C})$ and thus is modular. For this reason, this ${}_A\mathcal{C}$ (which is non-trivial if and only if $\mathcal{Z}_2(\mathcal{C}) \neq \mathcal{C}$, i.e., $\mathcal{C}$ is not symmetric) is called the modularization of $\mathcal{C}$, cf. [Bru], [Mü1].

For the purposes of this appendix, the case $\mathcal{S} \not\subset \mathcal{Z}_2(\mathcal{C})$ is more interesting:

3.7 Theorem. *If $\mathcal{C}$ is a braided fusion category, $\mathcal{S} \simeq \mathrm{Rep}_k\, G$ a full monoidal subcategory and (A, m, η) the corresponding commutative étale algebra, then*

(i) *${}_A\mathcal{C}$ is a braided G-crossed fusion category, which we denote as $\mathcal{C} \rtimes \mathcal{S}$.*

(ii) $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$ *as braided fusion category.*

(iii) *If* $\mathcal{D}$ *is a braided* G*-crossed fusion category and* $\mathrm{Rep}\, G \simeq \mathcal{S} \subset \mathcal{D}^G$ *as in Section* 5.1 *then* $\mathcal{D}^G \rtimes \mathcal{S} \simeq \mathcal{D}$ *as braided* G*-crossed fusion category.*

The G-spectrum and the degree-zero subcategory of $\mathcal{C} \rtimes \mathcal{S}$ can be described quite explicitly:

3.8 Proposition. *Under the assumptions of Theorem* 3.7*, we have:*

(i) *The degree zero part of* $\mathcal{C} \rtimes \mathcal{S}$ *is given by* $(\mathcal{C} \rtimes \mathcal{S})_e = \mathcal{S}' \rtimes \mathcal{S} = {}_A\mathcal{C}^{\mathrm{loc}}$. *(Here* $\mathcal{S}' \subset \mathcal{C}$ *is the 'centralizer' of* $\mathcal{S}$*, i.e., the full subcategory of objects* X *such that* $c(X,Y) \circ c(Y,X) = \mathrm{id}_{X \otimes Y}$ *for all* $Y \in \mathcal{S}$*. In particular* $\mathcal{C}' = \mathcal{Z}_2(\mathcal{C})$*. Furthermore,* ${}_A\mathcal{C}^{\mathrm{loc}} \subset {}_A\mathcal{C}$ *is the full subcategory consisting of* A*-modules* (X,μ) *that are dyslexic or local, i.e., satisfy* $\mu \circ c(X,A) \circ c(A,X) = \mu$*. It is known* [Pa] *that* ${}_A\mathcal{C}^{\mathrm{loc}}$ *is always braided.) The braided category* $(\mathcal{C} \rtimes \mathcal{S})_e$ *is modular if and only if* $\mathcal{Z}_2(\mathcal{S}') = \mathcal{S}$.

(ii) *The* G*-spectrum of* $\mathcal{C} \rtimes \mathcal{S}$ *is given by*

$$\mathrm{Spec}(\mathcal{C} \rtimes \mathcal{S}) = \{g \in G \mid \pi(g) = \mathrm{id}_V \quad \text{for all } (V,\pi) \in \mathcal{S} \cap \mathcal{Z}_2(\mathcal{C})\},$$

where we use $\mathcal{S} \simeq \mathrm{Rep}_k\, G$*. In particular, the grading is trivial if and only if* $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$ *and full if and only if* $\mathcal{S} \cap \mathcal{Z}_2(\mathcal{C})$ *is trivial, i.e., contains only multiples of the identity.*

3.9 Remark. 1. In particular, if $\mathcal{C}$ is modular then $\mathcal{Z}_2(\mathcal{C})$ is trivial and thus $\mathcal{C} \rtimes \mathcal{S}$ has full G-spectrum for any $\mathcal{S}$. Furthermore, the double centralizer theorem [Mü3] gives $\mathcal{S}'' = \mathcal{S}$, thus $\mathcal{Z}(\mathcal{S}') := \mathcal{S}' \cap \mathcal{S}'' = \mathcal{S}' \cap \mathcal{S} = \mathcal{S}$ (since $\mathcal{S} \subset \mathcal{S}'$), and therefore $\mathcal{S}' \rtimes \mathcal{S}$ is modular.

2. The proofs are too long to be given here; cf. [Kir1], [Kir2], [Mü4]. We only remark that statement 4 of Proposition 3.2 is crucial for obtaining both the G-action on and the G-grading of $\mathcal{C} \rtimes \mathcal{S}$ and for showing that the natural candidate for a braiding (which really is a braiding when $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$) actually is a braiding in the G-crossed sense.

3. In [DGNO], the passage $\mathcal{C} \rightsquigarrow \mathcal{C} \rtimes \mathcal{S}$ is called 'de-equivariantization'.

The final unsurprising result shows that $\mathcal{C} \rtimes \mathcal{S}$ depends on $\mathcal{S} \subset \mathcal{C}$ only up to equivalence:

3.10 Proposition. *Let* $E: \mathcal{C} \to \mathcal{D}$ *be an equivalence of braided fusion categories and* $\mathrm{Rep}_k\, G \simeq \mathcal{S} \subset \mathcal{C}$ *a full braided monoidal subcategory. Then there is an equivalence* $\mathcal{C} \rtimes \mathcal{S} \simeq \mathcal{D} \rtimes E(\mathcal{S})$ *of braided crossed* G*-categories.*

The proof relies on the fact that the commutative étale algebra in $\mathrm{Rep}\, G$ corresponding to the regular representation of G is unique up to isomorphism.

5.4 Classification and coherence for braided crossed G-categories

Combining the results of the two preceding sections we arrive at the following result:

4.1 Theorem. (i) *The operations $\mathcal{D} \rightsquigarrow \mathcal{D}^G$ and $\mathcal{C} \rightsquigarrow \mathcal{C} \rtimes \mathcal{S}$ give rise to a bijection between {braided G-crossed fusion categories $\mathcal{D}$, modulo equivalence of braided G-crossed categories} and {braided fusion categories $\mathcal{C}$ containing a full symmetric subcategory $\mathcal{S} \simeq \operatorname{Rep} G$, modulo braided equivalence}.*

(ii) *Under this correspondence, $\mathcal{C} \simeq \mathcal{D}^G$ is modular if and only if $\mathcal{D}_e$ is modular and $\mathcal{D}$ has full G-spectrum.*

Proof. (i) is contained in the results of the preceding sections.

(ii) That modularity of $\mathcal{C}$ implies modularity of $(\mathcal{C} \rtimes \mathcal{S})_e$ and full G-spectrum of $\mathcal{C} \rtimes \mathcal{S}$ is contained in Theorem 3.7. As to the converse, let $\mathcal{D}$ have full G-spectrum and $\mathcal{D}_e$ be modular. Defining $\mathcal{C} = \mathcal{C}^G$ and $\mathcal{C}_0 = (\mathcal{D}_e)^G$ we have $\mathcal{C} \supset \mathcal{C}_0 \supset \mathcal{S} \simeq \operatorname{Rep} G$. Modularity of $\mathcal{D}_e \simeq \mathcal{C}_0 \rtimes \mathcal{S}$ implies

$$\mathcal{S} = \mathcal{Z}_2(\mathcal{C}_0) = \mathcal{C}_0 \cap \mathcal{C}_0'. \tag{4.1}$$

Since $\mathcal{C}_0 \subset \mathcal{C}$ is the maximal subcategory for which $\mathcal{C}_0 \rtimes \mathcal{S}$ has trivial grading, we have

$$\mathcal{C} \cap \mathcal{S}' = \mathcal{C}_0. \tag{4.2}$$

The fullness of the G spectrum of $\mathcal{D} \simeq \mathcal{C} \rtimes \mathcal{S}$ implies that

$$\mathcal{S} \cap \mathcal{Z}_2(\mathcal{C}) = \mathcal{S} \cap \mathcal{C}' \text{ is trivial.} \tag{4.3}$$

If now $X \in \mathcal{Z}_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$ is simple, (4.2) implies $X \in \mathcal{C}_0$, upon which (4.1) implies $X \in \mathcal{S}$. But now (4.3) entails that X is trivial. Thus $\mathcal{Z}_2(\mathcal{C})$ is trivial, to wit $\mathcal{C}$ is modular. □

4.2 Remark. 1. A more satisfactory statement of the above correspondence would be in terms of a 2-equivalence of certain bicategories, cf. [DGNO].

2. An interesting alternative characterization of braided G-crossed fusion categories $\mathcal{D}$ satisfying the two conditions of (ii) is given in [Kir2].

We are now in a position to obtain a straightforward but useful application, which shows that no restriction of generality is entailed by the limitation to G-categories with strict G-action:

4.3 Theorem. *Let G be a finite group and $((\mathcal{D}, \dots), \beta, \partial, c)$ a braided G-crossed fusion category. Then there is a strict braided fusion category $(\mathcal{D}', \beta', \partial', c')$ with a strict G-action and an equivalence $F : \mathcal{D} \to \mathcal{D}'$ of braided crossed G-categories. (Thus in particular, F is G-equivariant.)*

Proof. Given a braided G-crossed fusion category $\mathcal{D}$, we have an equivalence $\mathcal{D} \simeq \mathcal{D}^G \rtimes \mathcal{S}$ of braided crossed G-categories, where $\mathrm{Rep}_k\, G \simeq \mathcal{S} \subset \mathcal{D}^G$. By the coherence theorem for braided monoidal categories, there is a strict braided monoidal category $\widetilde{\mathcal{C}} \simeq \mathcal{D}^G$ with a distinguished strict symmetric subcategory $\widetilde{\mathcal{S}}$. By Proposition 3.10, we have $\mathcal{D} \simeq \widetilde{\mathcal{C}} \rtimes \widetilde{\mathcal{S}}$ as braided crossed G-categories. The claim now follows from the fact that there is a model for $\widetilde{\mathcal{C}} \rtimes \widetilde{\mathcal{S}}$ that is strict as a monoidal category and has a strict G-action. This is the category $\widetilde{\mathcal{C}}_A$ ($A \in \widetilde{\mathcal{S}}$ again being the left regular representation) discussed in [Mü1], [Mü4], where also the equivalence with ${}_A\mathcal{C}$ was proven. □

It would be quite interesting to prove the theorem in a more direct way, hopefully extending its domain of validity.

5.5 Braided crossed G-categories as crossed products

The axioms of a crossed G-category $\mathcal{D}$ imply that the part $\mathcal{D}_e$ in trivial degree is a monoidal category with G-action β. In the case where $\mathcal{D}$ is a fusion category one can prove that $\dim \mathcal{D}_g \equiv \sum_{X \in \mathcal{D}_g} d(X)^2 = \dim \mathcal{D}_e$ (the sum being over the classes of simple objects in $\mathcal{D}_g$), whenever $\mathcal{D}_g$ is non-trivial, see Chapter VII, Section 1.7. This makes it reasonable to consider $\mathcal{D}$ as a crossed product of $\mathcal{D}_e$ with G: "$\mathcal{D} \simeq \mathcal{D}_e \rtimes_\beta G$". The question therefore arises whether, given a monoidal category with G-action there exists a crossed G-category $\mathcal{D}$ with full G-spectrum (i.e., $\mathcal{D}_g \neq \emptyset$ for all g) and a G-equivariant equivalence $\mathcal{C} \to \mathcal{D}_e$. Similarly, if $\mathcal{C}$ is braided and β an action $G \curvearrowright \mathcal{C}$ by braided automorphisms, one can ask for $\mathcal{D}$ to be a braided crossed G-category.

In the non-braided case it is easy to give an affirmative answer, discovered independently in the preprint [Tu4], cf. Section 2.1 of Chapter VIII of this book, and in [Ta]. For simplicity of exposition, we assume $\mathcal{C}$ and the G-action to be strict. (As we know by Theorem 4.3, this is justified in the case of fusion categories, but everything can also be done without strictness assumptions; cf. in particular [Ga].) We define a monoidal category $\mathcal{D}$ by $\mathrm{Obj}\,\mathcal{D} = \mathrm{Obj}\,\mathcal{C} \times G$ with tensor product $(X, g) \otimes (Y, h) = (X \otimes {}^gY, gh)$. The Hom-set $\mathrm{Hom}_{\mathcal{D}}((X, g), (Y, h))$ is defined as $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ when $g = h$ and by $\emptyset$ or $\{0\}$ (in the k-linear case) for $g \neq h$, composition being inherited from $\mathcal{C}$. Finally, if $s \in \mathrm{Hom}((X, g), (X', g))$ and $t \in \mathrm{Hom}((Y, h), (Y', h))$ then $s \otimes t := s \otimes {}^gt \in \mathrm{Hom}_{\mathcal{C}}(X \otimes {}^gY, X' \otimes {}^gY') = \mathrm{Hom}_{\mathcal{D}}((X, g) \otimes (Y, h), (X', g) \otimes (Y', h))$. $\mathcal{D}$ has an obvious G-grading $\partial\colon (X, g) \mapsto g$ and G-action ${}^g(X, h) = ({}^gX, ghg^{-1})$ with respect to which it is a crossed G-category. (Notice that $\mathcal{C} \rtimes G$ is not closed under direct sums of objects of different degrees. While this can easily be remedied, this is not needed for the discussion that follows.) The question arises whether a braiding for $\mathcal{C}$ can be lifted to a braiding for $\mathcal{D} = \mathcal{C} \rtimes G$. The following simple observation in [Mü5] provides an obstruction:

5.1 Lemma. *Let $\mathcal{D}$ be a braided crossed G-category. If there exists an invertible object in $\mathcal{D}_g$ then ${}^gX \cong X$ for every $X \in \mathcal{D}_e$.*

Proof. Let $Y \in \mathcal{D}_g$ be invertible and $X \in \mathcal{D}_e$. The braiding of $\mathcal{D}$ provides isomorphisms $Y \otimes X \cong {}^gX \otimes Y \cong Y \otimes {}^gX$. Invertibility of Y implies $X \cong {}^gX$. □

Since $\mathcal{C} \rtimes G$ has invertible objects $(\mathbb{1}, g)$ for every $g \in G$, the lemma implies that the crossed G-category $\mathcal{C} \rtimes G$ can have a braiding only if $X \cong {}^gX$ for all $g \in G$, $X \in \mathcal{C}$. In many situations, this is an unacceptable restriction. Nevertheless, it is interesting to say a bit more about braidings on $\mathcal{C} \rtimes G$. The following result is essentially a converse of the construction of a braiding on $\mathcal{C} \rtimes G$ given in Theorem VIII.2.3.1.

5.2 Proposition. *A braiding on $\mathcal{C} \rtimes G$ gives rise to a full and faithful monoidal functor $F\colon \mathcal{C} \to \mathcal{C}^G$ such that $K \circ F = \mathrm{id}_{\mathcal{C}}$, where $K\colon \mathcal{C}^G \to \mathcal{C}$ is the forgetful functor $(X, \{u_g\}) \mapsto X$, and therefore to an identification of $\mathcal{C}$ with a full monoidal subcategory of $\mathcal{C}^G$.*

Proof. In $\mathcal{C} \rtimes G$, we have $(\mathbb{1}, g) \otimes (X, e) = ({}^gX, g)$ and $(X, e) \otimes (\mathbb{1}, e) = (X, g)$. Thus the braiding $c_{(\mathbb{1},g),(X,e)}\colon ({}^gX, g) \xrightarrow{\cong} (X, g)$ provides an isomorphism $u_{X,g}\colon {}^gX \to X$. The braid identity for $c_{(\mathbb{1},g)\otimes(\mathbb{1},h),(X,e)}$ implies that $\{u_{X,g}\}_{g\in G}$ satisfies (2.1), thus $(X, \{u_{X,g}\})$ is an object in $\mathcal{C}^G$. The naturality of the braiding c implies that $F\colon X \mapsto (X, \{u_{X,g}\})$ is a functor $\mathcal{C} \to \mathcal{C}^G$. This functor is faithful by construction, and it is full since by definition $\mathrm{Hom}_{\mathcal{C}^G}((X, \{u_g\}), (Y, \{v_g\})) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$. By construction, it is clear that $K \circ F = \mathrm{id}_{\mathcal{C}}$. Finally, the braid identity for $c_{(\mathbb{1},g),(X,e)\otimes(Y,e)}$ is equivalent to $u_{X\otimes Y,g} = u_{X,g} \otimes u_{Y,g}$, which implies that F is strict monoidal. □

Lemma 5.1 shows that the straightforward crossed product $\mathcal{C} \rtimes G$ in general cannot be equipped with a braiding. (For a much more extensive discussion of $\mathcal{C} \rtimes G$, including the non-strict case, and braidings on it, cf. [Ga].) In order to construct braided crossed G-categories, one needs to adopt a more sophisticated approach, starting from the observation that each category $\mathcal{E}_g$ is a bimodule category over $\mathcal{E}_e$. However, this is not the place to do so. Instead, we point out that the problem of defining *braided* crossed products $\mathcal{C} \rtimes G$ is closely related to one raised in an earlier conjecture of the author. In the remainder of this section, we assume the ground field to be $\mathbb{C}$, which implies $d(X)^2 \geq 0$ for every object X, cf. [ENO]. Thus, if $\mathcal{C} \subset \mathcal{D}$ is a full subcategory, we have $\dim \mathcal{C} \leq \dim \mathcal{D}$, and equality holds if and only if the categories are equivalent.

In [Mü3], it was proven that if $\mathcal{D}$ is a modular category and $\mathcal{C} \subset \mathcal{D}$ a full monoidal subcategory, then $\dim \mathcal{D} \geq \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C})$ holds. Thus, there is a lower bound on the size (as measured by the dimension) of a modular category containing a given braided fusion category as a full subcategory. Furthermore, the following general conjecture was formulated, motivated by situations where it is true:

5.3 Conjecture ([Mü4]). *Every braided fusion category $\mathcal{C}$ embeds fully into a modular category $\mathcal{D}$ with* $\dim \mathcal{D} = \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C})$.

Now one has:

5.4 Theorem. *The following are equivalent:*

(i) *Conjecture 5.3 holds for every braided fusion category $\mathcal{C}$ whose symmetric center $\mathcal{Z}_2(\mathcal{C})$ is even (and therefore equivalent to* Rep G *for a finite group G).*

(ii) *For every modular category $\mathcal{M}$ acted upon by a finite group G there is a braided crossed G-category $\mathcal{E}$ with full G-spectrum and a G-equivariant equivalence $\mathcal{E}_e \simeq \mathcal{M}$.*

Proof. (ii) $\Longrightarrow$ (i): Let $\mathcal{C}$ be a braided fusion category with even center. By the reconstruction theorem, there is a finite group G such that $\mathcal{S} = \mathcal{Z}_2(\mathcal{C}) \simeq \mathrm{Rep}_k\, G$. Being the modularization [Bru], [Mü1] of $\mathcal{C}$, $\mathcal{M} = \mathcal{C} \rtimes \mathcal{S}$ is modular, and it carries a G-action such that $\mathcal{M}^G \simeq \mathcal{C}$. By assumption (i), there is a braided crossed G-category $\mathcal{E}$ with full G-spectrum and G-equivariant equivalence $\mathcal{E}_e \simeq \mathcal{M}$. This implies $\dim \mathcal{E} = |G| \dim \mathcal{M} = \dim \mathcal{C}$. Now $\mathcal{D} = \mathcal{E}^G$ is a braided fusion category containing $\mathcal{M}^G \simeq \mathcal{C}$ as a full braided subcategory. We have $\dim \mathcal{D} = |G| \dim \mathcal{E} = |G| \dim \mathcal{C} = \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(G)$. Finally, $\mathcal{D}$ is modular by Theorem 4.1 (ii).

(i) $\Longrightarrow$ (ii): Let $\mathcal{M}$ be a modular category with G-action. Then $\mathcal{C} = \mathcal{M}^G$ is a braided fusion category with $\mathcal{S} \simeq \mathrm{Rep}_k\, G$ as full braided subcategory. Since $\mathcal{M} \simeq \mathcal{C} \rtimes \mathcal{S}$ has trivial G-grading and is modular, we have $\mathcal{Z}_2(\mathcal{C}) = \mathcal{S}$. By assumption (ii), there is a full braided embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ with $\mathcal{D}$ modular of dimension $\dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C}) = |G| \dim \mathcal{C} = |G|^2 \dim \mathcal{M}$. In view of $\mathcal{S} \subset \mathcal{C} \subset \mathcal{D}$, we can consider the braided crossed G-category $\mathcal{E} = \mathcal{D} \rtimes \mathcal{S}$. Since $\mathcal{D}$ is modular, $\mathcal{E}$ has full G-spectrum, and we have the G-equivariant equivalence $\mathcal{E}_e = (\mathcal{D} \cap \mathcal{S}') \rtimes \mathcal{S} = \mathcal{C} \rtimes \mathcal{S} \simeq \mathcal{M}$. □

The significance of this result is that the problem of minimal embeddings into modular categories, for which no direct approach is in sight, can be reduced to the crossed product problem which appears more amenable, if by no means easy. However it seems that this problem does not always have a solution: According to V. Ostrik and collaborators (private communication concerning as-yet-unpublished work), there exists a cohomological obstruction. However this may turn out, there is a special situation where there is reason to believe that the mentioned obstruction vanishes:

5.5 Conjecture. *Let $\mathcal{C}$ be a modular category and N a positive integer. Then there exists a distinguished braided crossed G-category $\mathcal{C} \wr S_N$ with $G = S_N$, full spectrum and an equivalence $(\mathcal{C} \wr S_N)_e \simeq \mathcal{C}^{\boxtimes N}$ that is equivariant with respect to the obvious S_N-action on $\mathcal{C}^{\boxtimes N}$.*

Unfortunately, no good formulation in terms of a universal property is known. However, there is a hypothetical application to quantum field theory, which we will state at the end of the next section.

5.6 Remarks on applications in conformal field theory

In this concluding section, we briefly outline the connection of the results described in this appendix to conformal field theory. In fact, much of the author's work was motivated by such applications, and the same is probably true of [Kir1], [Kir2]. Since it is not possible to go into technical details, we limit ourselves to indicating the broad line of ideas and giving some pertinent references. A complete account can be found in [Mü5].

In 1971, Doplicher, Haag and Roberts (DHR) studied [DHR] a class of representations of a quantum field theory defined on $3+1$ dimensional Minkowski space in the context of the operator algebraic approach to axiomatic quantum field theory. (The latter had been founded by Haag and others around 1960. For a recent review of some aspects, cf. [Ha].) DHR showed that the category of the representations under consideration is a rigid semisimple unitary symmetric monoidal category, and they conjectured that such a category always is equivalent to the representation category of a compact (super)group. In the late 1980s, this was proven by Doplicher and Roberts [DR1], and Deligne independently arrived at an analogous result [Del] for pro-algebraic groups. (He did not require the categories to be unitary.) Doplicher and Roberts also proved [DR2] that, given a quantum field theory $\mathcal{A}$ and the compact group G such that $\mathrm{Rep}\,\mathcal{A} \simeq \mathrm{Rep}\,G$, there exists an extended quantum field theory $\mathcal{F}$ acted upon by G such that $\mathcal{F}^G \cong \mathcal{A}$ and such that $\mathrm{Rep}\,\mathcal{F}$ is trivial (at least when G is second countable and $\mathrm{Rep}\,\mathcal{A}$ is even, i.e., the supergroup is a group). Thus, the existence of non-trivial representations of a quantum field theory $\mathcal{A}$ can be understood as a consequence of $\mathcal{A}$ being the G-fixed subtheory $\mathcal{F}^G$ of some 'bigger' quantum field theory $\mathcal{F}$. Furthermore, there is a one-to-one correspondence between quantum field theories $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$ and closed subgroups $H \subset G$, given by $H \mapsto \mathcal{F}^H$ and

$$\mathcal{B} \mapsto H^{\mathcal{B}} = \{g \in G \mid g \restriction \mathcal{B} = \mathrm{id}\}.$$

These results amount to a beautiful Galois theory for local fields, where the Doplicher–Roberts extension $\mathcal{F}$ corresponds to the algebraic closure. As a consequence of this theory, the representation categories of the extensions $\mathcal{B} \supset \mathcal{A}$ can be understood in purely group theoretic terms, without any rôle for the dynamics of the quantum fields.

All the results described above remain valid in $2+1$ spacetime dimensions, but in $1+1$ dimensions (or on 'the lightray' $\mathbb{R}$) the situation changes considerably. As shown in [FRS], a quantum field theory still gives rise to a rigid semisimple unitary braided monoidal category of representations, but one can no more prove that the braiding is a symmetry, quite in line with the theoretical physics literature. A host of 'rational' models studied in conformal field theory suggested that the representation category $\mathrm{Rep}\,\mathcal{A}$ should be a modular category under suitable assumptions. In [KLM], a very simple and natural set of axioms for a chiral conformal field theory, known to be satisfied by several infinite families of interesting examples associated with loop groups, was shown to imply modularity of the representation category. (A similar result was also proven in the context of vertex operator algebras, cf. [Hu].) Since a non-trivial modular category cannot be the representation category of a group, it is clear that analogues of the results of [DR1], [DR2] cannot be expected. Not even replacing the group by a more general algebraic structure, e.g. Hopf algebra, is very promising; cf. [Mü7] for a discussion of this issue.

While groups thus lose the distinguished rôle they played in higher spacetime dimensions, it is perfectly natural to study group actions on conformal field theories and the corresponding fixpoint theories $\mathcal{F}^G$, called 'orbifold theories'. One classical result of DHR remains true, namely the representation category $\mathrm{Rep}\,\mathcal{F}^G$ still contains a full symmetric monoidal subcategory equivalent to $\mathrm{Rep}\,G$. On the other hand, given a full symmetric subcategory $\mathcal{S} \subset \mathrm{Rep}\,\mathcal{A}$, the construction in [DR2] applies and provides an extension $\mathcal{F} = \mathcal{A} \rtimes \mathcal{S} \supset \mathcal{A}$ acted upon by the compact group dual to the symmetric category $\mathcal{S}$. As in high dimensions, the passages $\mathcal{A} \rightsquigarrow \mathcal{A} \rtimes \mathcal{S}$ to the extended theory and the orbifolding $\mathcal{F} \rightsquigarrow \mathcal{F}^G$ are essentially inverses of each other, i.e., $(\mathcal{A} \rtimes \mathcal{S})^G \cong \mathcal{A}$ and $\mathcal{F}^G \rtimes \mathcal{S} \cong \mathcal{F}$.

However, the relationship between the categories $\mathrm{Rep}\,\mathcal{A}$ and $\mathrm{Rep}\,\mathcal{A}^G$ on the one hand and between $\mathrm{Rep}\,\mathcal{A}$ and $\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S})$ will be more complicated than in the high dimensional situation. In the context of completely rational conformal field theories [KLM], it was shown in [Mü5] that the representation category of an extension $\mathcal{A} \rtimes \mathcal{S}$ of a quantum field theory $\mathcal{A}$ by a symmetric subcategory of $\mathrm{Rep}\,\mathcal{A}$ is given by

$$\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S}) \simeq (\mathrm{Rep}(\mathcal{A}) \cap \mathcal{S}') \rtimes \mathcal{S}. \tag{6.1}$$

(Here as in all that follows, we must assume that G is finite. Otherwise the theories $\mathcal{A}$ and $\mathcal{A} \rtimes \mathcal{S}$ cannot both be completely rational.) While we saw in Section 5.3 that a braided fusion category $\mathcal{C}$ can be recovered from its extension $\mathcal{C} \rtimes \mathcal{S}$ by a full symmetric subcategory via $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$, there is little reason to hope that $\mathrm{Rep}\,\mathcal{A}$ can be recovered from $\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S})$, since this is given by (6.1) and some information is lost in the passage from $\mathrm{Rep}\,\mathcal{A}$ to $\mathrm{Rep}(\mathcal{A}) \cap \mathcal{S}'$. Since the Doplicher–Roberts construction and orbifolding are inverse operations, it follows that also the representation category $\mathrm{Rep}\,\mathcal{F}^G$ of an orbifold quantum field theory is not determined by $\mathrm{Rep}\,\mathcal{F}$. This was already understood in the early works on orbifolds, e.g. [DVVV]. (This phenomenon can be seen even in the simplest case, the one where $\mathrm{Rep}\,\mathcal{F}$ is trivial, i.e., equivalent to $\mathrm{Vect}_{\mathbb{C}}$. In this specific situation, it turns out that $\mathrm{Rep}\,\mathcal{F}^G \simeq D^\omega(G)$-Mod, where $D^\omega(G)$ is the twisted quantum double. The cohomology class $[\omega] \in H^3(G, U(1))$ is encoded in $\mathcal{F}$, but clearly not in the trivial category $\mathrm{Rep}\,\mathcal{F}$.)

The solution to the problem of computing $\mathrm{Rep}\,\mathcal{F}^G$ in terms of categorical information pertaining to $\mathcal{F}$ was found in [Mü5]:

6.1 Theorem. *To a completely rational conformal field theory $\mathcal{F}$ acted upon freely by a finite group G, one can associate a braided crossed G-category G-$\mathrm{Rep}\,\mathcal{F}$ with full G-spectrum. The degree zero subcategory is the category of ordinary representations as considered by Doplicher, Haag and Roberts* [DHR] (*known to be modular by* [KLM]).

The non-trivially graded objects correspond to 'twisted representations' of $\mathcal{F}$. While they do not satisfy the DHR criterion, they do so upon restriction to the orbifold theory $\mathcal{F}^G$, which explains their relevance for the determination of $\mathrm{Rep}(\mathcal{F}^G)$. With these preparations, one can prove [Mü5]:

6.2 Theorem. *If $\mathcal{F}$ is a completely rational CFT carrying a free action of a finite group G, then there is an equivalence*

$$\mathrm{Rep}(\mathcal{F}^G) \simeq (G\text{-}\mathrm{Rep}\,\mathcal{F})^G$$

of braided monoidal categories. Conversely, one has the equivalence

$$G\text{-}\mathrm{Rep}\,\mathcal{F} \simeq \mathrm{Rep}(\mathcal{F}^G) \rtimes \mathcal{S}$$

of braided crossed G-categories, where $\mathcal{S} \subset \mathrm{Rep}(\mathcal{F}^G)$ is the symmetric subcategory of representations of $\mathcal{F}^G$ arising from the vacuum representations of $\mathcal{F}$.

Thus, the braided G-crossed category of twisted representations of $\mathcal{F}$ and the representation category $\mathrm{Rep}\,\mathcal{F}^G$, together with its symmetric subcategory $\mathcal{S} \subset \mathrm{Rep}\,\mathcal{F}^G$, contain the same information. In particular, this clarifies the phenomenon [DVVV] that $\mathrm{Rep}\,\mathcal{F}$ does not determine $\mathrm{Rep}\,\mathcal{F}^G$.

Now we are in a position to connect Conjecture 5.5 to conformal field theory. Let $\mathcal{A}$ be a completely rational conformal field theory and N a positive integer. Now the N-fold direct tensor power $\mathcal{B} = \mathcal{A}^{\boxtimes N}$ of $\mathcal{A}$ carries an obvious S_N-action and it is natural to conjecture the following:

6.3 Conjecture. *If $\mathcal{A}$ is a completely rational CFT* (*thus* $\mathrm{Rep}\,\mathcal{A}$ *is modular*) *then the* (*braided S_N-crossed*) *category of S_N-twisted representations of $\mathcal{A}^{\boxtimes N}$ is equivalent to the category* $(\mathrm{Rep}\,A) \wr S_N$ *of Conjecture* 5.5. (*Thus* S_N-$\mathrm{Rep}(\mathcal{A}^{\boxtimes N})$ *depends on* $\mathcal{A}$ *only via* $\mathrm{Rep}\,\mathcal{A}$.)

This conjecture is compatible with the rigorous work that has been done on CFTs of the form $\mathcal{A}^{\boxtimes N}$ and the fixpoint theories ('orbifolds') $(\mathcal{A}^{\boxtimes N})^G$ for $G \subset S_N$, cf. e.g. [BDM], [KLX], but to our understanding the Conjectures 5.5 and 6.3 are still open.

Appendix 6
Algebraic properties of Hopf G-coalgebras
by Alexis Virelizier

Let G be a group. The notion of a (ribbon) Hopf G-coalgebra was first introduced by Turaev [Tu4], as the prototype algebraic structure whose category of representations is a (ribbon) G-category (see Section VIII.1). Recall from Chapter VII that ribbon G-categories give rise to invariants of 3-dimensional G-manifolds and to 3-dimensional HQFTs with target $K(G,1)$. Moreover, Hopf G-coalgebras may be used directly (without involving their representations) to construct further topological invariants of 3-dimensional G-manifolds, see Appendix 7.

Here we review the algebraic properties of Hopf G-coalgebras and provide examples. Most of the results are given without proof, see [Vir1]–[Vir4] for details.

In Section 1, we study the algebraic properties of Hopf G-coalgebras, in particular the existence of integrals, the order of the antipode (a generalization of the Radford S^4-formula), and the (co)semisimplicity (a generalization of the Maschke theorem).

In Section 2, we focus on quasitriangular and ribbon Hopf G-coalgebras. In particular we construct G-traces for ribbon Hopf G-coalgebras, which are used to construct invariants of 3-dimensional G-manifolds in Appendix 7.

In Section 3, we give a method for constructing a quasitriangular Hopf G-coalgebra starting from a Hopf algebra endowed with an action of G by Hopf automorphisms. This leads to non-trivial examples of quasitriangular Hopf G-coalgebras for all finite G and for some infinite G such as $\mathrm{GL}_n(K)$. In particular, we define graded quantum groups.

Throughout this appendix, G is a group (with neutral element 1) and K is a field. All algebras are supposed to be over K, associative, and unital. The tensor product $\otimes = \otimes_K$ of K-vector spaces is always taken over K. If U and V are K-vector spaces, then $\sigma_{U,V} \colon U \otimes V \to V \otimes U$ denotes the flip defined by $\sigma_{U,V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$.

6.1 Hopf G-coalgebras

1.1 Hopf G-coalgebras. We recall, for completeness, the definition of a Hopf G-coalgebra from Section VIII.1, but with a minor change: we do not suppose the antipode to be bijective.

A *Hopf G-coalgebra* (over K) is a family $H = \{H_\alpha\}_{\alpha \in G}$ of K-algebras endowed with a family $\Delta = \{\Delta_{\alpha,\beta} \colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta\}_{\alpha,\beta \in G}$ of algebra homomorphisms (the *comultiplication*), an algebra homomorphism $\varepsilon\colon H_1 \to K$ (the *counit*), and a family $S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in G}$ of K-linear maps (the *antipode*) such that, for all $\alpha, \beta, \gamma \in G$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

$$(\mathrm{id}_{H_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{id}_{H_\alpha} = (\varepsilon \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha},$$

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon\, 1_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}},$$

where $m_\alpha\colon H_\alpha \otimes H_\alpha \to H_\alpha$ and $1_\alpha \in H_\alpha$ denote multiplication in H_α and the unit element of H_α.

When $G = 1$, one recovers the usual notion of a Hopf algebra. In particular, H_1 is a Hopf algebra.

Remark that the notion of a Hopf G-coalgebra is not self-dual (the dual notion obtained by reversing the arrows in the definition may be called a Hopf G-algebra).

If $H = \{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra, then the set $\{\alpha \in G \mid H_\alpha \neq 0\}$ is a subgroup of G. Also, if G' is a subgroup of G, then $H = \{H_\alpha\}_{\alpha\in G'}$ is a Hopf G'-coalgebra.

The antipode S of a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is anti-multiplicative (in the sense that each $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ is an anti-homomorphism of algebras) and anti-comultiplicative in the sense that $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ for all $\alpha, \beta \in G$ and $\varepsilon S_1 = \varepsilon$; see [Vir2], Lemma 1.1.

A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is said to be of *finite type* if, for all $\alpha \in G$, H_α is finite-dimensional (over K). Note that the direct sum $\bigoplus_{\alpha\in G} H_\alpha$ is finite-dimensional if and only if H is of finite type and $H_\alpha = 0$ for all but a finite number of $\alpha \in G$.

The antipode $S = \{S_\alpha\}_{\alpha\in G}$ of $H = \{H_\alpha\}_{\alpha\in G}$ is said to be *bijective* if each S_α is bijective. Unlike in Section VIII.1, we do not suppose that the antipode of a Hopf G-coalgebra is bijective. As for Hopf algebras, the antipode of a Hopf G-coalgebra H is necessarily bijective if H is of finite type (see Section 1.5) or H is quasitriangular (see Section 2.4).

1.2 The case of finite G. Suppose that G is a finite group. Recall that the Hopf algebra K^G of functions on G has a basis $(e_\alpha\colon G \to K)_{\alpha\in G}$ defined by $e_\alpha(\beta) = \delta_{\alpha,\beta}$ where $\delta_{\alpha,\alpha} = 1$ and $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$. The structure maps of K^G are given by

$$e_\alpha e_\beta = \delta_{\alpha,\beta}\, e_\alpha, \quad 1_{K^G} = \sum_{\alpha\in G} e_\alpha, \quad \Delta(e_\alpha) = \sum_{\beta\gamma=\alpha} e_\beta \otimes e_\gamma, \quad \varepsilon(e_\alpha) = \delta_{\alpha,1},$$

and $S(e_\alpha) = e_{\alpha^{-1}}$. A *central prolongation* of K^G is a Hopf algebra A endowed with a morphism of Hopf algebras $K^G \to A$, called the *central map*, which carries K^G into the center of A.

Since G is finite, any Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ gives rise to a Hopf algebra $\widetilde{H} = \bigoplus_{\alpha\in G} H_\alpha$ with structure maps given by

$$\widetilde{\Delta}|_{H_\alpha} = \sum_{\beta\gamma=\alpha} \Delta_{\beta,\gamma}, \quad \tilde{\varepsilon}|_{H_\alpha} = \delta_{\alpha,1}\,\varepsilon, \quad \widetilde{m}|_{H_\alpha\otimes H_\beta} = \delta_{\alpha,\beta}\, m_\alpha, \quad \tilde{1} = \sum_{\alpha\in G} 1_\alpha,$$

and $\widetilde{S} = \sum_{\alpha\in G} S_\alpha$. The K-linear map $K^G \to \widetilde{H}$ defined by $e_\alpha \mapsto 1_\alpha$ gives rise to a morphism of Hopf algebras which carries K^G into the center of $\widetilde{H}$. Hence $\widetilde{H}$ is a central prolongation of K^G.

The correspondence assigning to every Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ the central prolongation $K^G \to \widetilde{H}$ is bijective. Given a Hopf algebra $(A, m, 1, \Delta, \varepsilon, S)$ which is a central prolongation of K^G, set $H_\alpha = A1_\alpha$, where $1_\alpha \in A$ is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$. Then the family $\{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra with structure maps given by

$$m_\alpha = 1_\alpha \cdot m|_{H_\alpha\otimes H_\alpha}, \quad \Delta_{\alpha,\beta} = (1_\alpha \otimes 1_\beta)\cdot \Delta|_{H_{\alpha\beta}}, \quad \varepsilon = \varepsilon|_{H_1}, \quad S_\alpha = 1_{\alpha^{-1}} \cdot S|_{H_\alpha}.$$

1.3 Integrals. Recall that a left (resp. right) integral for a Hopf algebra $(A, \Delta, \varepsilon, S)$ is an element $\Lambda \in A$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra A^* is a K-linear form $\lambda \in A^* = \mathrm{Hom}_K(A, K)$ such that $(\mathrm{id}_A \otimes \lambda)\Delta(x) = \lambda(x)1_A$ (resp. $(\lambda \otimes \mathrm{id}_A)\Delta(x) = \lambda(x)1_A$) for all $x \in A$.

A *left* (resp. *right*) *G-integral* for a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is a family of K-linear forms $\lambda = (\lambda_\alpha)_{\alpha\in G} \in \prod_{\alpha\in G} H_\alpha^*$ such that

$$(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\alpha \ \text{(resp. } (\lambda_\alpha \otimes \mathrm{id}_{H_\beta})\Delta_{\alpha,\beta}(x) = \lambda_{\alpha\beta}(x)1_\beta\text{)}$$

for all $\alpha, \beta \in G$ and $x \in H_{\alpha\beta}$. Note that λ_1 is a usual left (resp. right) integral for the Hopf algebra H_1^*.

A G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ is said to be *non-zero* if $\lambda_\beta \neq 0$ for some $\beta \in G$. Given a non-zero G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$, we have $\lambda_\alpha \neq 0$ for all $\alpha \in G$ such that $H_\alpha \neq 0$. In particular $\lambda_1 \neq 0$.

It is known that the K-vector space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one-dimensional. This extends to Hopf G-coalgebras as follows.

Theorem A ([Vir2], Theorem 3.6). *Let H be a Hopf G-coalgebra of finite type. Then the vector space of left* (*resp. right*) *G-integrals for H is one-dimensional.*

The proof of this theorem is based on the fact that a Hopf G-comodule has a canonical decomposition generalizing the fundamental decomposition theorem in the theory of Hopf modules.

1.4 Grouplike elements. A family $g = (g_\alpha)_{\alpha\in G} \in \Pi_{\alpha\in G} H_\alpha$ such that $\Delta_{\alpha,\beta}(g_{\alpha\beta}) = g_\alpha \otimes g_\beta$ for all $\alpha, \beta \in G$ and $\varepsilon(g_1) = 1_K$ is called a *G-grouplike element* of a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$. Note that g_1 is then a grouplike element of the Hopf algebra H_1 in the usual sense of the word.

One easily checks that the set $\mathrm{Gr}(H)$ of G-grouplike elements of H is a group with respect to coordinate-wise multiplication in the product monoid $\Pi_{\alpha\in G} H_\alpha$. If $g = (g_\alpha)_{\alpha\in G} \in \mathrm{Gr}(H)$, then $g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha\in G}$. The group $\mathrm{Hom}(G, K^*)$ of homomorphisms $G \to K^*$ acts on $\mathrm{Gr}(H)$ by $\phi g = (\phi(\alpha) g_\alpha)_{\alpha\in G}$ for arbitrary $\phi \in \mathrm{Hom}(G, K^*)$ and $g = (g_\alpha)_{\alpha\in G} \in \mathrm{Gr}(H)$.

1.5 The distinguished G-grouplike element. Throughout this subsection, $H = \{H_\alpha\}_{\alpha\in G}$ is a Hopf G-coalgebra of finite type with antipode $S = \{S_\alpha\}_{\alpha\in G}$. Using Theorem A, one verifies that there is a unique G-grouplike element $g = (g_\alpha)_{\alpha\in G}$ of H, called the *distinguished G-grouplike element of H*, such that $(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta}\, g_\alpha$ for any right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ and all $\alpha, \beta \in G$. Note that g_1 is the distinguished grouplike element of H_1.

Since H_1 is a finite-dimensional Hopf algebra, there exists a unique algebra morphism $\nu\colon H_1 \to K$ such that if Λ is a left integral for H_1, then $\Lambda x = \nu(x)\Lambda$ for all $x \in H_1$. This morphism is a grouplike element of the Hopf algebra H_1^*, called the *distinguished grouplike element of* H_1^*. It is invertible in H_1^* and its inverse ν^{-1} is also an algebra morphism. Moreover, if Λ is a right integral for H_1, then $x\Lambda = \nu^{-1}(x)\Lambda$ for all $x \in H_1$.

For all $\alpha \in G$, we define a left and a right H_1^*-action on H_α by setting, for all $f \in H_1^*$ and $a \in H_\alpha$,

$$f \rightharpoonup a = (\mathrm{id}_{H_\alpha} \otimes f)\Delta_{\alpha,1}(a) \quad \text{and} \quad a \leftharpoonup f = (f \otimes \mathrm{id}_{H_\alpha})\Delta_{1,\alpha}(a).$$

The next assertion generalizes Theorem 3 of [Rad3]. This is a key result in the theory of Hopf G-coalgebras. It is used in particular to prove the existence of traces (see Section 2.8).

Theorem B ([Vir2], Theorem 4.2). *Let $\lambda = (\lambda_\alpha)_{\alpha\in G}$ be a right G-integral for H. Then, for all $\alpha \in G$ and $x, y \in H_\alpha$,*

(a) $\lambda_\alpha(xy) = \lambda_\alpha(S_{\alpha^{-1}} S_\alpha(y \leftharpoonup \nu)\, x)$;
(b) $\lambda_\alpha(xy) = \lambda_\alpha(y\, S_{\alpha^{-1}} S_\alpha(\nu^{-1} \rightharpoonup g_\alpha^{-1} x g_\alpha))$;
(c) $\lambda_{\alpha^{-1}}(S_\alpha(x)) = \lambda_\alpha(g_\alpha x)$.

As a corollary we obtain a generalization of the celebrated Radford S^4-formula to Hopf G-coalgebras:

Corollary C ([Vir2], Lemma 4.6). *Let $H = \{H_\alpha\}_{\alpha\in G}$ be a Hopf G-coalgebra of finite type. Then for all $\alpha \in G$ and $x \in H_\alpha$,*

$$(S_{\alpha^{-1}} S_\alpha)^2(x) = g_\alpha(\nu \rightharpoonup x \leftharpoonup \nu^{-1})g_\alpha^{-1}.$$

This formula implies in particular that the antipode S of H is bijective (i.e., each S_α is bijective).

1.6 The order of the antipode. It is known that the order of the antipode of a finite-dimensional Hopf algebra is finite ([Rad1], Theorem 1) and divides four times the dimension of the algebra ([NZ], Proposition 3.1). We apply this result to study a Hopf G-coalgebra of finite type $H = \{H_\alpha\}_{\alpha\in G}$ with antipode $S = \{S_\alpha\}_{\alpha\in G}$. Let α be an element of G of finite order d. Denote by $\langle\alpha\rangle$ the subgroup of G generated by α. By considering the finite-dimensional Hopf algebra $\bigoplus_{\beta\in\langle\alpha\rangle} H_\beta$ (determined by the Hopf $\langle\alpha\rangle$-coalgebra $\{H_\beta\}_{\beta\in\langle\alpha\rangle}$, see Section 1.2), we obtain that the order of $S_{\alpha^{-1}}S_\alpha \in \mathrm{Aut}_{\mathrm{Alg}}(H_\alpha)$ is finite and divides $2\sum_{\beta\in\langle\alpha\rangle} \dim H_\beta$. From Corollary C, we obtain another upper bound on the order of $S_{\alpha^{-1}}S_\alpha$: if $\alpha \in G$ has a finite order d, then the order of $S_{\alpha^{-1}}S_\alpha$ divides $2d \dim H_1$; see [Vir2], Corollary 4.5.

1.7 Semisimplicity. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is said to be *semisimple* if each algebra H_α is semisimple. For H to be semisimple it is necessary that H_1 be finite-dimensional (since an infinite-dimensional Hopf algebra over a field is not semisimple, see [Sw], Corollary 2.7). When H is of finite type, H is semisimple if and only if H_1 is semisimple, see [Vir2], Lemma 5.1.

1.8 Cosemisimplicity. The notion of a comodule over a coalgebra may be extended to the setting of Hopf G-coalgebras. A *right G-comodule* over a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ is a family $M = \{M_\alpha\}_{\alpha\in G}$ of K-vector spaces endowed with a family of K-linear maps

$$\rho = \{\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes H_\beta\}_{\alpha,\beta\in G}$$

such that

$$(\rho_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})\rho_{\alpha\beta,\gamma} = (\mathrm{id}_{M_\alpha} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma} \quad \text{and} \quad (\mathrm{id}_{M_\alpha} \otimes \varepsilon)\rho_{\alpha,1} = \mathrm{id}_{M_\alpha}$$

for all $\alpha, \beta, \gamma \in G$. A *$G$-subcomodule* of M is a family $N = \{N_\alpha\}_{\alpha\in G}$, where N_α is a K-subspace of M_α, such that $\rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes H_\beta$ for all $\alpha, \beta \in G$. The sums and direct sums for families of G-subcomodules of a right G-comodule are defined in the obvious way.

A right G-comodule $M = \{M_\alpha\}_{\alpha\in G}$ is said to be *simple* if it is *non-zero* (i.e., $M_\alpha \neq 0$ for some $\alpha \in G$) and if it has no G-subcomodules other than itself and the trivial one $0 = \{0\}_{\alpha\in G}$. A right G-comodule which is a direct sum of a family of simple G-subcomodules is said to be *cosemisimple*. Note that all G-subcomodules and all quotients of a cosemisimple right G-comodule are cosemisimple.

A Hopf G-coalgebra is *cosemisimple* if it is cosemisimple as a right G-comodule over itself (with comultiplication as comodule map). By [Vir2], a Hopf G-coalgebra

$H = \{H_\alpha\}_{\alpha \in G}$ is cosemisimple if and only if every reduced[1] right G-comodule over H is cosemisimple.

We state a Hopf G-coalgebra version of the dual Maschke theorem.

Theorem D ([Vir2], Theorem 5.4). *A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is cosemisimple if and only if there exists a right G-integral $\lambda = (\lambda_\alpha)_{\alpha \in G}$ for H such that $\lambda_\alpha(1_\alpha) = 1_K$ for some $\alpha \in G$ (and then $\lambda_\alpha(1_\alpha) = 1_K$ for all $\alpha \in G$ with $H_\alpha \neq 0$).*

As corollaries, we obtain that a Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ of finite type is cosemisimple if and only if the Hopf algebra H_1 is cosemisimple, and that the distinguished G-grouplike element of a cosemisimple Hopf G-coalgebra of finite type is trivial.

1.9 Involutory Hopf G-coalgebras. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is *involutory* if its antipode $S = \{S_\alpha\}_{\alpha \in \pi}$ satisfies the identity $S_{\alpha^{-1}} S_\alpha = \mathrm{id}_{H_\alpha}$ for all $\alpha \in \pi$.

Involutory Hopf G-coalgebras of finite type have special properties. For example, each of their G-integrals $\lambda = (\lambda_\alpha)_{\alpha \in G}$ is two sided, S-invariant ($\lambda_{\alpha^{-1}} S_\alpha = \lambda_\alpha$ for all $\alpha \in G$), and symmetric ($\lambda_\alpha(xy) = \lambda_\alpha(yx)$ for all $\alpha \in G$ and $x, y \in H_\alpha$). Also if the ground field K of H is of characteristic 0, then $\dim H_\alpha = \dim H_1$ whenever $H_\alpha \neq 0$.

Finally, if $H = \{H_\alpha\}_{\alpha \in G}$ is an involutory Hopf G-coalgebra of finite type over a field whose characteristic does not divide $\dim H_1$, then H is semisimple and cosemisimple; see [Vir4], Lemma 3.

6.2 Quasitriangular Hopf G-coalgebras

2.1 Crossed Hopf G-coalgebras. A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is *crossed* if it is endowed with a *crossing*, that is, a family of algebra isomorphisms $\varphi = \{\varphi_\beta \colon H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta \in G}$ such that

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta, \quad \varepsilon\varphi_\beta = \varepsilon, \quad \text{and} \quad \varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta$$

for all $\alpha, \beta, \gamma \in G$. One easily verifies that a crossing preserves the antipode, that is, $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ for all $\alpha, \beta \in G$. Therefore this definition of a crossed Hopf G-coalgebra is equivalent to the one in Chapter VIII.

A crossing φ in H yields a group homomorphism $\varphi \colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(H_1)$ and determines thus an action of G on H_1 by Hopf algebra automorphisms. Here for a Hopf algebra A, we denote $\mathrm{Aut}_{\mathrm{Hopf}}(A)$ the group of Hopf automorphisms of A.

If G is an abelian group, then any Hopf G-coalgebra admits a *trivial crossing* $\varphi_\beta = \mathrm{id}$ for all $\beta \in G$.

When G is a finite group, the notion of a crossing can be described in terms of central prolongations of K^G (see Section 1.2): a *crossing* of a central prolongation A

[1] A right G-comodule $M = \{M_\alpha\}_{\alpha \in G}$ over H is *reduced* if $M_\alpha = 0$ whenever $H_\alpha = 0$.

of K^G is a group homomorphism $\varphi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ such that $\varphi_\beta(1_\alpha) = 1_{\beta\alpha\beta^{-1}}$ for all $\alpha, \beta \in G$, where 1_α is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$.

2.2 The distinguished character. Let $H = \{H_\alpha\}_{\alpha\in G}$ be a crossed Hopf G-coalgebra of finite type with crossing φ. Using the uniqueness of G-integrals (see Theorem A), one can show the existence of a unique group homomorphism $\widehat{\varphi}\colon G \to K^*$, called the *distinguished character of* H, such that $\lambda_{\beta\alpha\beta^{-1}}\varphi_\beta = \widehat{\varphi}(\beta)\,\lambda_\alpha$ for any left or right G-integral $\lambda = (\lambda_\alpha)_{\alpha\in G}$ for H and all $\alpha, \beta \in G$.

Lemma E ([Vir2], Lemma 6.3). *For any $\beta \in G$,*

(a) *If Λ is a left or right integral for H_1, then $\varphi_\beta(\Lambda) = \widehat{\varphi}(\beta)\Lambda$.*
(b) *If ν is the distinguished grouplike element of H_1^*, then $\nu\varphi_\beta = \nu$.*
(c) *If $g = (g_\alpha)_{\alpha\in G}$ is the distinguished G-grouplike element of H, then $\varphi_\beta(g_\alpha) = g_{\beta\alpha\beta^{-1}}$ for all $\alpha \in G$.*

2.3 Quasitriangular Hopf G-coalgebras. Following Chapter VIII, we call a crossed Hopf G-coalgebra $(H = \{H_\alpha\}_{\alpha\in G}, \varphi)$ *quasitriangular* if it is endowed with an *R-matrix*, that is, a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in G}$ of invertible elements such that, for all $\alpha, \beta, \gamma \in G$ and $x \in H_{\alpha\beta}$,

$$\begin{aligned}
R_{\alpha,\beta}\cdot\Delta_{\alpha,\beta}(x) &= \sigma_{\beta,\alpha}(\varphi_{\alpha^{-1}}\otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x)\cdot R_{\alpha,\beta},\\
(\mathrm{id}_{H_\alpha}\otimes\Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) &= (R_{\alpha,\gamma})_{1\beta 3}\cdot(R_{\alpha,\beta})_{12\gamma},\\
(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma}) &= [(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3}\cdot(R_{\beta,\gamma})_{\alpha 23},\\
(\varphi_\beta\otimes\varphi_\beta)(R_{\alpha,\gamma}) &= R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}.
\end{aligned}$$

Here $\sigma_{\beta,\alpha}$ denotes the flip $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$ and, for K-vector spaces P, Q and $r = \sum_j p_j \otimes q_j \in P \otimes Q$, we set

$$r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma, \quad r_{\alpha 23} = 1_\alpha \otimes r \in H_\alpha \otimes P \otimes Q,$$

and $r_{1\beta 3} = \sum_j p_j \otimes 1_\beta \otimes q_j \in P \otimes H_\beta \otimes Q$. Note that $R_{1,1}$ is an R-matrix for the Hopf algebra H_1 is the usual sense of the word.

When G is abelian and φ is the trivial crossing, we recover the definition of a quasitriangular G-colored Hopf algebra due to Ohtsuki [Oh1].

An R-matrix for a crossed Hopf G-coalgebra provides a solution of the G-colored Yang–Baxter equation

$$\begin{aligned}
&(R_{\beta,\gamma})_{\alpha 23}\cdot(R_{\alpha,\gamma})_{1\beta 3}\cdot(R_{\alpha,\beta})_{12\gamma}\\
&\qquad = (R_{\alpha,\beta})_{12\gamma}\cdot[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3}\cdot(R_{\beta,\gamma})_{\alpha 23}
\end{aligned}$$

and satisfies the following identities (see [Vir2], Lemma 6.4): for all $\alpha, \beta, \gamma \in G$,

$$(\varepsilon \otimes \mathrm{id}_{H_\alpha})(R_{1,\alpha}) = 1_\alpha = (\mathrm{id}_{H_\alpha} \otimes \varepsilon)(R_{\alpha,1}),$$
$$(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\beta})(R_{\alpha^{-1},\beta}) = R_{\alpha,\beta}^{-1} \quad \text{and} \quad (\mathrm{id}_{H_\alpha} \otimes S_\beta)(R_{\alpha,\beta}^{-1}) = R_{\alpha,\beta^{-1}},$$
$$(S_\alpha \otimes S_\beta)(R_{\alpha,\beta}) = (\varphi_\alpha \otimes \mathrm{id}_{H_{\beta^{-1}}})(R_{\alpha^{-1},\beta^{-1}}).$$

2.4 The Drinfeld element. The *Drinfeld element* of a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is the family $u = (u_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} H_\alpha$, where

$$u_\alpha = m_\alpha(S_{\alpha^{-1}}\varphi_\alpha \otimes \mathrm{id}_{H_\alpha})\,\sigma_{\alpha,\alpha^{-1}}(R_{\alpha,\alpha^{-1}}).$$

Observe that u_1 is the Drinfeld element of the quasitriangular Hopf algebra H_1 (see [Drin]). By [Vir2], Lemma 6.5, each u_α is invertible in H_α and

$$u_\alpha^{-1} = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}}S_\alpha)\,\sigma_{\alpha,\alpha}(R_{\alpha,\alpha}).$$

Moreover, for any $\alpha \in G$ and $x \in H$,

$$S_{\alpha^{-1}}S_\alpha(x) = u_\alpha \varphi_{\alpha^{-1}}(x) u_\alpha^{-1},$$

where φ is the crossing in H. This implies that the antipode of H is bijective.

Note also the identities $\varepsilon(u_1) = 1$, $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$, and

$$\Delta_{\alpha,\beta}(u_{\alpha\beta}) = [\sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}]^{-1} \cdot (u_\alpha \otimes u_\beta).$$

2.5 Ribbon Hopf G-coalgebras. Following Chapter VIII, we call a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ *ribbon* if it is endowed with a *twist*, that is, a family of invertible elements $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha \in G}$ such that for all $\alpha, \beta \in G$ and $x \in H_\alpha$,

$$\varphi_\alpha(x) = \theta_\alpha^{-1} x \theta_\alpha, \quad S_\alpha(\theta_\alpha) = \theta_{\alpha^{-1}}, \quad \varphi_\beta(\theta_\alpha) = \theta_{\beta\alpha\beta^{-1}},$$
$$\Delta_{\alpha,\beta}(\theta_{\alpha\beta}) = (\theta_\alpha \otimes \theta_\beta) \cdot \sigma_{\beta,\alpha}(\mathrm{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}.$$

Note that θ_1 is a twist of the quasitriangular Hopf algebra H_1, and so $\varepsilon(\theta_1) = 1$. If $\alpha \in G$ has a finite order d, then θ_α^d is a central element of H_α. In particular, θ_1 is central in H_1.

Example. Let G be a group and $c \colon G \times G \to K^*$ be a bicharacter of G, that is, $c(\alpha, \beta\gamma) = c(\alpha,\beta)\,c(\alpha,\gamma)$ and $c(\alpha\beta,\gamma) = c(\alpha,\gamma)\,c(\beta,\gamma)$ for all $\alpha, \beta, \gamma \in G$. Consider the following crossed Hopf algebra K^c: for all $\alpha, \beta \in G$, we have $K_\alpha^c = K$ as an algebra and

$$\Delta_{\alpha,\beta}(1_K) = 1_K \otimes 1_K, \quad \varepsilon(1_K) = 1_K, \quad S_\alpha(1_K) = 1_K, \quad \varphi_\beta(1_K) = 1_K.$$

Then K^c is a ribbon Hopf G-coalgebra of finite type with R-matrix and twist given by $R_{\alpha,\beta} = c(\alpha,\beta)\,1_K \otimes 1_K$ and $\theta_\alpha = c(\alpha,\alpha)$. The Drinfeld elements of K^c are computed by $u_\alpha = c(\alpha,\alpha)^{-1}$.

2.6 The spherical G-grouplike element. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a ribbon Hopf G-coalgebra with Drinfeld element $u = (u_\alpha)_{\alpha \in G}$. For any $\alpha \in G$, set

$$w_\alpha = \theta_\alpha u_\alpha = u_\alpha \theta_\alpha \in H_\alpha.$$

Then $w = (w_\alpha)_{\alpha \in G}$ is a G-grouplike element, called the *spherical G-grouplike element of H*. It satisfies the identities

$$\varphi_\beta(w_\alpha) = w_{\beta\alpha\beta^{-1}}, \quad S_\alpha(u_\alpha) = w_{\alpha^{-1}}^{-1} u_{\alpha^{-1}} w_{\alpha^{-1}}^{-1}, \quad \text{and} \quad S_{\alpha^{-1}} S_\alpha(x) = w_\alpha x w_\alpha^{-1}$$

for all $\alpha, \beta \in G$ and $x \in H_\alpha$. Conversely, any G-grouplike element $w = (w_\alpha)_{\alpha \in G}$ of a quasitriangular Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ which satisfies these identities gives rise to a twist $\theta = (\theta_\alpha)_{\alpha \in G}$ in H by $\theta_\alpha = w_\alpha u_\alpha^{-1} = u_\alpha^{-1} w_\alpha$.

2.7 Further G-grouplike elements. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a quasitriangular Hopf G-coalgebra of finite type. Set

$$\ell_\alpha = S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} u_\alpha = u_\alpha\, S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} \in H_\alpha,$$

where $u = (u_\alpha)_{\alpha \in G}$ is the Drinfeld element of H. The properties of u ensure that $\ell = (\ell_\alpha)_{\alpha \in G}$ is a G-grouplike element of H. Let ν be the distinguished grouplike element of H_1^* and $\widehat{\varphi}$ be the distinguished character of H (see Sections 1.5 and 2.2). Denoting $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta \in G}$ the R-matrix of H, set

$$h_\alpha = (\mathrm{id}_{H_\alpha} \otimes \nu)(R_{\alpha,1}) \in H_\alpha.$$

Theorem F ([Vir2], Theorem 6.9). *The family $h = (h_\alpha)_{\alpha \in G}$ is a G-grouplike element of H. The distinguished G-grouplike element $(g_\alpha)_{\alpha \in G}$ of H is computed by $g_\alpha = \widehat{\varphi}(\alpha)^{-1} \ell_\alpha h_\alpha$ for all $\alpha \in G$.*

For ribbon H, we obtain as a corollary that $g_\alpha = \widehat{\varphi}(\alpha)^{-1} w_\alpha^2 h_\alpha$ for all $\alpha \in G$, where $w = (w_\alpha)_{\alpha \in G}$ is the spherical G-grouplike element of H.

2.8 Traces. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a crossed Hopf G-coalgebra. A *G-trace* for H is a family of K-linear forms $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha \in G} \in \Pi_{\alpha \in G} H_\alpha^*$ such that

$$\mathrm{tr}_\alpha(xy) = \mathrm{tr}_\alpha(yx), \quad \mathrm{tr}_{\alpha^{-1}}(S_\alpha(x)) = \mathrm{tr}_\alpha(x), \quad \text{and} \quad \mathrm{tr}_{\beta\alpha\beta^{-1}}(\varphi_\beta(x)) = \mathrm{tr}_\alpha(x)$$

for all $\alpha, \beta \in G$ and $x, y \in H_\alpha$. Note that tr_1 is a usual trace for the Hopf algebra H_1, which is invariant under the action φ of G.

A Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is *unimodular* if the Hopf algebra H_1 is unimodular (that is the spaces of left and right integrals for H_1 coincide). If H_1 is finite-dimensional, then H is unimodular if and only if $\nu = \varepsilon$, where ν is the distinguished grouplike element of H_1^*. For example, any finite type semisimple Hopf G-coalgebra is unimodular.

Consider in more detail a unimodular ribbon Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$ of finite type. Let $\lambda = (\lambda_\alpha)_{\alpha\in G}$ be a non-zero right G-integral for H, $w = (w_\alpha)_{\alpha\in G}$ be the spherical G-grouplike element of H, and $\widehat{\varphi}$ be the distinguished character of H.

Using Theorems B and F, we obtain that the G-traces for H are parameterized by families $z = (z_\alpha)_{\alpha\in G}$ such that $z_\alpha \in H_\alpha$ is central, $S_\alpha(z_\alpha) = \widehat{\varphi}(\alpha)^{-1} z_{\alpha^{-1}}$, and $\varphi_\beta(z_\alpha) = \widehat{\varphi}(\beta) z_{\beta\alpha\beta^{-1}}$ for all $\alpha, \beta \in G$. The G-trace corresponding to such a family z is given by $\mathrm{tr}_\alpha(x) = \lambda_\alpha(w_\alpha z_\alpha x)$. We point out two such families.

Let Λ be a left integral for H_1 such that $\lambda_1(\Lambda) = 1$. Set $z_1 = \Lambda$ and $z_\alpha = 0$ if $\alpha \neq 1$. The resulting family $(z_\alpha)_{\alpha\in G}$ satisfies all the conditions above since H is unimodular (and so Λ is central and $S_1(\Lambda) = \Lambda$) and by Lemma E (a). The corresponding G-trace is given by $\mathrm{tr}_1 = \varepsilon$ and $\mathrm{tr}_\alpha = 0$ for all $\alpha \neq 1$.

If $\widehat{\varphi}(\alpha) = 1$ for all $\alpha \in G$, then another possible choice of a family z is $z_\alpha = 1_\alpha$ for all α. Note that $\widehat{\varphi} = 1$ if H is semisimple or cosemisimple or if $\lambda_1(\theta_1) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha\in G}$ is the twist of H. We obtain the following assertion.

Theorem G ([Vir2], Theorem 7.4). *Suppose under the assumptions above that at least one of the following four conditions is satisfied: H is semisimple, or H is cosemisimple, or $\lambda_1(\theta_1) \neq 0$, or $\varphi_\beta|_{H_1} = \mathrm{id}_{H_1}$ for all $\beta \in G$. Then the family of K-linear maps* $\mathrm{tr} = (\mathrm{tr}_\alpha)_{\alpha\in G}$, *defined by* $\mathrm{tr}_\alpha(x) = \lambda_\alpha(w_\alpha x)$ *for all $x \in H_\alpha$, is a G-trace for H.*

6.3 The twisted double construction

Starting from a crossed Hopf G-coalgebra $H = \{H_\alpha\}_{\alpha\in G}$, Zunino [Zu1] constructed a double $Z(H) = \{Z(H)_\alpha\}_{\alpha\in G}$ of H which is a quasitriangular Hopf G-coalgebra containing H as a Hopf G-subcoalgebra. As a vector space, $Z(H)_\alpha = H_\alpha \otimes (\bigoplus_{\beta\in G} H^*_\beta)$. Generally speaking, $Z(H)$ is not of finite type: the components $Z(H)_\alpha$ may be infinite-dimensional.

In this section we provide a method, called the twisted double construction, for deriving quasitriangular Hopf G-coalgebras of finite type from finite-dimensional Hopf algebras endowed with action of G by Hopf automorphisms (cf. Section 2.1). We also give an h-adic version of this construction. This will lead us to non-trivial examples of quasitriangular Hopf G-coalgebras for any finite group G and for infinite groups G such as $\mathrm{GL}_n(K)$. In particular, we define the (h-adic) graded quantum groups.

3.1 Hopf pairings. Recall that a *Hopf pairing* between two Hopf algebras A and B (over K) is a bilinear pairing $\sigma\colon A \times B \to K$ such that, for all $a, a' \in A$ and $b, b' \in B$,

$$\sigma(a, bb') = \sigma(a_{(1)}, b)\,\sigma(a_{(2)}, b'), \quad \sigma(a, 1) = \varepsilon(a),$$
$$\sigma(aa', b) = \sigma(a, b_{(2)})\,\sigma(a', b_{(1)}), \quad \sigma(1, b) = \varepsilon(b).$$

Such a pairing always preserves the antipode: $\sigma(S(a), S(b)) = \sigma(a, b)$ for all $a \in A$ and $b \in B$.

A Hopf pairing $\sigma\colon A \times B \to K$ determines two annihilator ideals $I_A = \{a \in A \mid \sigma(a,b) = 0 \text{ for all } b \in B\}$ and $I_B = \{b \in B \mid \sigma(a,b) = 0 \text{ for all } a \in A\}$. It is easy to check that I_A and I_B are Hopf ideals of A and B, respectively. The pairing σ is *non-degenerate* iff $I_A = I_B = 0$. Any Hopf pairing $\sigma\colon A \times B \to K$ induces a non-degenerate Hopf pairing $\bar{\sigma}\colon A/I_A \times B/I_B \to K$.

3.2 The twisted double. Let $\sigma\colon A \times B \to K$ be a Hopf pairing between two Hopf algebras A and B, and let $\phi\colon A \to A$ be a Hopf algebra endomorphism of A. Set

$$D(A,B;\sigma,\phi) = A \otimes B$$

as a K-vector space. We provide $D(A,B;\sigma,\phi)$ with a structure of an algebra with unit $1_A \otimes 1_B$ and multiplication

$$(a \otimes b)\cdot(a' \otimes b') = \sigma(\phi(a'_{(1)}), S(b_{(1)}))\,\sigma(a'_{(3)}, b_{(3)})\; aa'_{(2)} \otimes b_{(2)}b'$$

for any $a, a' \in A$ and $b, b' \in B$.

Note that if ϕ and ϕ' are different Hopf algebra endomorphisms of A, then the algebras $D(A,B;\sigma,\phi)$ and $D(A,B;\sigma,\phi')$ are in general not isomorphic (see Remark in Section 3.4 below).

Theorem H ([Vir3], Theorem 2.6). *Let $\sigma\colon A \times B \to K$ be a Hopf pairing between Hopf algebras A and B, and let ϕ be an action of G on A by Hopf algebra automorphisms, that is, ϕ is a group homomorphism $G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$. Then the family of algebras*

$$D(A,B;\sigma,\phi) = \{D(A,B;\sigma,\phi_\alpha)\}_{\alpha \in G}$$

has a structure of a Hopf G-coalgebra given by

$$\begin{aligned}
\Delta_{\alpha,\beta}(a \otimes b) &= (\phi_\beta(a_{(1)}) \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}),\\
\varepsilon(a \otimes b) &= \varepsilon_A(a)\,\varepsilon_B(b),\\
S_\alpha(a \otimes b) &= \sigma(\phi_\alpha(a_{(1)}), b_{(1)})\,\sigma(a_{(3)}, S(b_{(3)}))\;\phi_\alpha S(a_{(2)}) \otimes S(b_{(2)})
\end{aligned}$$

for all $a \in A$, $b \in B$ and $\alpha, \beta \in G$. Furthermore, if σ is non-degenerate and A, B are finite dimensional, then the Hopf G-coalgebra $D(A,B;\sigma,\phi)$ is quasitriangular with crossing φ and R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in G}$ given by

$$\varphi_\beta(a \otimes b) = \phi_\beta(a) \otimes \phi^*_\beta(b) \quad \text{and} \quad R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i),$$

where $\phi^*\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(B)$ is the unique action such that $\sigma(\phi_\beta, \phi^*_\beta) = \sigma$ for all $\beta \in G$, and $(e_i)_i$ and $(f_i)_i$ are dual bases of A and B with respect to σ.

Corollary I. *Let A be a finite-dimensional Hopf algebra and ϕ be an action of G on A by Hopf algebra automorphisms. Then the duality bracket $\langle\,,\rangle_{A\otimes A^*}$ is a non-degenerate Hopf pairing between A and $A^{*\mathrm{cop}}$ and $D(A, A^{*\mathrm{cop}}; \langle\,,\rangle_{A\otimes A^*}, \phi)$ is a quasitriangular Hopf G-coalgebra.*

Note that the group of Hopf automorphisms of a finite-dimensional semisimple Hopf algebra A over a field of characteristic 0 is finite (see [Rad2]). To obtain quasitriangular Hopf G-coalgebras with infinite G using the twisted double method, one has to start from non-semisimple Hopf algebras or from Hopf algebras over fields of non-zero characteristic.

In the next three sections, we use Theorem H to give examples of quasitriangular Hopf G-coalgebras.

3.3 Example: finite G. Let G be a finite group. In this section, we describe the ribbon Hopf G-coalgebras obtained by the twisted double construction from the group algebra $K[G]$. The standard Hopf algebra structure on $K[G]$ is given by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$. The dual of $K[G]$ is the Hopf algebra $F(G) = K^G$ of functions $G \to K$ with structure maps and basis $(e_g : G \to K)_{g \in G}$ described in Section 2.1. Let $\phi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(K[G])$ be the homomorphism defined by $\phi_\alpha(h) = \alpha h \alpha^{-1}$ for $\alpha \in G, h \in K[G]$. Corollary I yields a quasitriangular Hopf G-coalgebra

$$D_G(G) = D(K[G], F(G)^{\mathrm{cop}}; \langle\, , \rangle_{K[G]\times F(G)}, \phi).$$

Let us describe $D_G(G) = \{D_\alpha(G)\}_{\alpha\in G}$ more precisely. For $\alpha \in G$, the algebra $D_\alpha(G)$ is equal to $K[G] \otimes F(G)$ as a K-vector space, has unit $1_{D_\alpha(G)} = \sum_{g\in G} 1 \otimes e_g$ and multiplication

$$(g \otimes e_h) \cdot (g' \otimes e_{h'}) = \delta_{\alpha g'\alpha^{-1}, h^{-1}g'h'}\, gg' \otimes e_{h'}$$

for all $g, g', h, h' \in G$. The structure maps of $D_G(G)$ are

$$\Delta_{\alpha,\beta}(g \otimes e_h) = \sum_{xy=h} \beta g \beta^{-1} \otimes e_y \otimes g \otimes e_x, \quad \varepsilon(g \otimes e_h) = \delta_{h,1},$$

$$S_\alpha(g \otimes e_h) = \alpha g^{-1}\alpha^{-1} \otimes e_{\alpha g \alpha^{-1}h^{-1}g^{-1}}, \quad \varphi_\alpha(g \otimes e_h) = \alpha g \alpha^{-1} \otimes e_{\alpha h \alpha^{-1}}$$

for all $\alpha, \beta, g, h \in G$. The crossed Hopf G-coalgebra $D_G(G)$ is quasitriangular and furthermore ribbon with R-matrix and twist

$$R_{\alpha,\beta} = \sum_{g,h\in G} g \otimes e_h \otimes 1 \otimes e_g \quad \text{and} \quad \theta_\alpha = \sum_{g\in G} \alpha^{-1} g \alpha \otimes e_g$$

for all $\alpha, \beta \in G$. The spherical G-grouplike element of $D_G(G)$ is $w = (1_{D_\alpha(G)})_{\alpha\in G}$. The family $\lambda = (\lambda_\alpha)_{\alpha\in G}$, defined by $\lambda_\alpha(g \otimes e_h) = \delta_{g,1}$, is a two-sided G-integral for $D_G(G)$.

3.4 An example of a quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra. In this section, K is a field of characteristic $\neq 2$ and n is a positive integer. Let A be the K-algebra with generators $g, x_1, \ldots, x_n$ subject to the relations

$$g^2 = 1, \quad x_i^2 = 0, \quad g x_i = -x_i g, \quad x_i x_j = -x_j x_i.$$

The algebra A is 2^{n+1}-dimensional and has a Hopf algebra structure given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \varepsilon(x_i) = 0, \quad S(g) = g,$$

and $S(x_i) = gx_i$ for all i. The group of Hopf automorphisms of A is isomorphic to the group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in K (see [Rad2]). An explicit isomorphism $\phi\colon \mathrm{GL}_n(K) \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ carries any $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(K)$ to the automorphism ϕ_α of A given by

$$\phi_\alpha(g) = g \quad \text{and} \quad \phi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, x_k.$$

We apply Corollary I to these A and ϕ. Observing that $A^* \cong A$ as Hopf algebras, we can quotient the resulting quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra to eliminate one copy of the generator g (which appears twice), see [Vir3], Proposition 4.1. This gives a quasitriangular Hopf $\mathrm{GL}_n(K)$-coalgebra $\mathcal{H} = \{\mathcal{H}_\alpha\}_{\alpha \in \mathrm{GL}_n(K)}$. We give here a direct description of $\mathcal{H}$. For $\alpha = (\alpha_{i,j}) \in \mathrm{GL}_n(K)$, let $\mathcal{H}_\alpha$ be the K-algebra generated g, $x_1, \dots x_n, y_1, \dots, y_n$, subject to the relations

$$g^2 = 1, \quad x_1^2 = \cdots = x_n^2 = 0, \quad gx_i = -x_i g, \quad x_i x_j = -x_j x_i,$$
$$y_1^2 = \cdots = y_n^2 = 0, \quad gy_i = -y_i g, \quad y_i y_j = -y_j y_i,$$
$$x_i y_j - y_j x_i = (\alpha_{j,i} - \delta_{i,j})\, g,$$

where $1 \le i, j \le n$. The family $\mathcal{H} = \{\mathcal{H}_\alpha\}_{\alpha \in \mathrm{GL}_n(K)}$ has the following structure of a crossed Hopf $\mathrm{GL}_n(K)$-coalgebra:

$$\Delta_{\alpha,\beta}(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S_\alpha(g) = g,$$
$$\Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} \beta_{k,i}\, x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, g x_k,$$
$$\Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_\alpha(y_i) = -g y_i,$$
$$\varphi_\alpha(g) = g, \quad \varphi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i}\, x_k, \quad \varphi_\alpha(y_i) = \sum_{k=1}^{n} \tilde{\alpha}_{i,k}\, y_k,$$

where $\alpha = (\alpha_{i,j})$, $\beta = (\beta_{i,j})$ run over $\mathrm{GL}_n(K)$, $(\tilde{\alpha}_{i,j}) = \alpha^{-1}$, and $1 \le i \le n$. The crossed Hopf $\mathrm{GL}_n(K)$-coalgebra $\mathcal{H}$ is quasitriangular with R-matrix

$$R_{\alpha,\beta} = \frac{1}{2} \sum_{S \subseteq \{1,\dots n\}} x_S \otimes y_S + x_S \otimes g y_S + g x_S \otimes y_S - g x_S \otimes g y_S$$

for all $\alpha, \beta \in \mathrm{GL}_n(K)$. Here $x_\emptyset = 1$, $y_\emptyset = 1$, and for a nonempty subset S of $\{1, \dots n\}$, we set $x_S = x_{i_1} \dots x_{i_s}$ and $y_S = y_{i_1} \dots y_{i_s}$, where $i_1 < \cdots < i_s$ are the elements of S.

Remark. Generally speaking, for distinct $\alpha, \beta \in \mathrm{GL}_n(K)$, the algebras $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ are not isomorphic. For example, $\mathcal{H}_\alpha \not\simeq \mathcal{H}_1$ for any $\alpha \in \mathrm{GL}_n(K) - \{1\}$. It suffices to prove that

$$\mathcal{H}_\alpha/[\mathcal{H}_\alpha, \mathcal{H}_\alpha] \not\simeq \mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1].$$

Indeed, $\mathcal{H}_\alpha/[\mathcal{H}_\alpha, \mathcal{H}_\alpha] = 0$ since $g = \frac{1}{\alpha_{j,i}-\delta_{i,j}}(x_i y_j - y_j x_i) \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$ (for some $1 \leq i, j \leq n$ such that $\alpha_{j,i} \neq \delta_{i,j}$) and so $1 = g^2 \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha]$. In $\mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1]$, we have $x_k = x_k g^2 = 0$ (since $x_k g = g x_k = -x_k g$ and so $x_k g = 0$) and likewise $y_k = 0$. Hence $\mathcal{H}_1/[\mathcal{H}_1, \mathcal{H}_1] = K\langle g \mid g^2 = 1\rangle \neq 0$.

3.5 Graded quantum groups. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. Let $\{d_i\}_{i=1}^l$ be coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric. Let q be a fixed non-zero complex number and $q_i = q^{d_i}$ for $i = 1, 2, \ldots, l$. We suppose that $q_i^2 \neq 1$ for all i.

Recall that the (usual) quantum group $U_q(\mathfrak{g})$ can be obtained as a quotient of the quantum double of $U_q(\mathfrak{b}_+)$, where $\mathfrak{b}_+$ is the (positive) Borel subalgebra of $\mathfrak{g}$ (the quotient is needed to eliminate the second copy of the Cartan subalgebra). Applying Theorem H to the Hopf algebra $U_q(\mathfrak{b}_+)$ endowed with an action of $(\mathbb{C}^*)^l$ by Hopf automorphisms, we obtain the "graded quantum group" introduced in [Vir3], Proposition 5.1. It can be directly described as follows.

Set $G = (\mathbb{C}^*)^l$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, let $U_q^\alpha(\mathfrak{g})$ be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}, E_i, F_i, 1 \leq i \leq l$, subject to the following defining relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_i E_j = q_i^{a_{i,j}} E_j K_i,$$
$$K_i F_j = q_i^{-a_{i,j}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\alpha_i K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad \text{if } i \neq j,$$
$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad \text{if } i \neq j.$$

The family $U_q^G(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha \in G}$ has a structure of a crossed Hopf G-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, $\beta = (\beta_1, \ldots, \beta_l) \in G$ and $1 \leq i \leq l$, by:

$$\Delta_{\alpha,\beta}(K_i) = K_i \otimes K_i,$$
$$\Delta_{\alpha,\beta}(E_i) = \beta_i E_i \otimes K_i + 1 \otimes E_i,$$
$$\Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$
$$S_\alpha(K_i) = K_i^{-1}, \quad S_\alpha(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_\alpha(F_i) = -K_i F_i,$$
$$\varphi_\alpha(K_i) = K_i, \quad \varphi_\alpha(E_i) = \alpha_i E_i, \quad \varphi_\alpha(F_i) = \alpha_i^{-1} F_i.$$

Note that $(U_q^1(\mathfrak{g}), \Delta_{1,1}, \varepsilon, S_1)$ is the usual quantum group $U_q(\mathfrak{g})$.

To give a rigorous treatment of R-matrices for the graded quantum groups, we need h-adic versions of Hopf G-coalgebras and of graded quantum groups. This is the content of the next two sections.

3.6 The *h*-adic case. In this section, we develop an h-adic variant of Hopf G-coalgebras. Roughly speaking, h-adic Hopf G-coalgebras are obtained by taking the ring $\mathbb{C}[[h]]$ of formal power series as the ground ring and requiring that the algebras (resp. the tensor products) are complete (resp. completed) in the h-adic topology.

Recall that if V is a (left) module over $\mathbb{C}[[h]]$, then the topology on V for which the sets $\{h^n V + v \mid n \in \mathbb{N}\}$ form a base for neighborhoods of $v \in V$ is called the *h-adic topology*. For $\mathbb{C}[[h]]$-modules V and W, denote by $V \hat{\otimes} W$ the completion of $V \otimes_{\mathbb{C}[[h]]} W$ in the h-adic topology.

If V is a complex vector space, then the set $V[[h]]$ of all formal power series $\sum_{n=0}^{\infty} v_n h^n$ with coefficients $v_n \in V$ is a $\mathbb{C}[[h]]$-module called a *topologically free module*. Topologically free modules are exactly $\mathbb{C}[[h]]$-modules which are complete, separated, and torsion-free. Furthermore, $V[[h]] \hat{\otimes} W[[h]] = (V \otimes W)[[h]]$ for any complex vector spaces V and W.

An *h-adic algebra* A is a $\mathbb{C}[[h]]$-module complete in the h-adic topology and endowed with a $\mathbb{C}[[h]]$-linear map $m\colon A \hat{\otimes} A \to A$ and an element $1 \in A$ such that $m(\mathrm{id}_A \hat{\otimes} m) = m(m \hat{\otimes} \mathrm{id}_A)$ and $m(\mathrm{id}_A \hat{\otimes} 1) = \mathrm{id}_A = m(1 \hat{\otimes} \mathrm{id}_A)$.

By an *h-adic Hopf G-coalgebra*, we mean a family $H = \{H_\alpha\}_{\alpha \in G}$ of h-adic algebras endowed with h-adic algebra homomorphisms $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \hat{\otimes} H_\beta$ $(\alpha, \beta \in G)$, $\varepsilon\colon A \to \mathbb{C}[[h]]$, and with $C[[h]]$-linear maps $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ $(\alpha \in G)$ satisfying formulas of Section 1.1. It is understood that the algebraic tensor product $\otimes$ is replaced everywhere by its h-adic completions $\hat{\otimes}$.

The notions of crossed, quasitriangular, and ribbon h-adic Hopf G-coalgebras can be defined similarly following Sections 2.1 and 2.3.

Theorem H carries over to the h-adic Hopf algebras. The key modifications are that $\sigma\colon A \hat{\otimes} B \to \mathbb{C}[[h]]$ must be $\mathbb{C}[[h]]$-linear and $D(A, B; \sigma, \phi) = A \hat{\otimes} B$.

Theorem J. *Let $\sigma\colon A \hat{\otimes} B \to \mathbb{C}[[h]]$ be an h-adic Hopf pairing between two h-adic Hopf algebras A and B. Let $\phi\colon G \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ be an action of G on A by h-adic Hopf automorphisms. Then the family $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha \in G}$ is an h-adic Hopf G-coalgebra. Assume furthermore that A and B are topologically free, σ is non-degenerate, and $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ belongs to the h-adic completion $D(A, B; \sigma, \phi_\alpha) \hat{\otimes} D(A, B; \sigma, \phi_\beta)$, where $(e_i)_i$ and $(f_i)_i$ are bases of A and B dual with respect to σ. Then $D(A, B; \sigma, \phi)$ is quasitriangular with R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in G}$.*

The condition on $R_{\alpha,\beta}$ in the second part of the theorem means the following. Since A and B are topologically free, $A = V[[h]]$ and $B = W[[h]]$ for some complex vector spaces V and W. Then

$$D(A, B; \sigma, \phi_\alpha) \hat{\otimes} D(A, B; \sigma, \phi_\beta) = (V \otimes W \otimes V \otimes W)[[h]].$$

We require that $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ can be expanded as $\sum_{n=0}^{\infty} r_n h^n$ for some $r_n \in V \otimes W \otimes V \otimes W$.

In the next section, we use Theorem J to define h-adic graded quantum groups.

3.7 *h*-adic graded quantum groups. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l with Cartan matrix $(a_{i,j})$. Let $\{d_i\}_{i=1}^{l}$ be coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric. Applying Theorem J to the h-adic Hopf algebras $U_h(\mathfrak{b}_+)$ and $\widetilde{U}_h(\mathfrak{b}_-) = \mathbb{C}[[h]]1 + hU_h(\mathfrak{b}_-)$, we obtain (after appropriate quotienting) quasitriangular "h-adic graded quantum groups" (see [Vir3], Proposition 6.1). We give here a direct description of these quantum groups.

Let $G = \mathbb{C}[[h]]^l$ with group operation being addition. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, let $U_h^\alpha(\mathfrak{g})$ be the h-adic algebra generated by the elements H_i, E_i, F_i, $1 \le i \le l$, subject to the following defining relations:

$$\begin{aligned}
[H_i, H_j] &= 0,\\
[H_i, E_j] &= a_{ij} E_j,\\
[H_i, F_j] &= -a_{ij} F_j,\\
[E_i, F_j] &= \delta_{i,j} \frac{e^{d_i h \alpha_i} e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}},
\end{aligned}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad (i \neq j),$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad (i \neq j).$$

The family $U_h^G(\mathfrak{g}) = \{U_h^\alpha(\mathfrak{g})\}_{\alpha \in G}$ has a structure of a crossed h-adic Hopf G-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_l) \in G$ and $1 \le i \le l$, by

$$\begin{aligned}
\Delta_{\alpha,\beta}(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0,\\
\Delta_{\alpha,\beta}(E_i) &= e^{d_i h \beta_i} E_i \otimes e^{d_i h H_i} + 1 \otimes E_i, \quad \varepsilon(E_i) = 0,\\
\Delta_{\alpha,\beta}(F_i) &= F_i \otimes 1 + e^{-d_i h H_i} \otimes F_i, \quad \varepsilon(F_i) = 0,
\end{aligned}$$

$$S_\alpha(H_i) = -H_i, \quad S_\alpha(E_i) = -e^{d_i h \alpha_i} E_i e^{-d_i h H_i}, \quad S_\alpha(F_i) = -e^{d_i h H_i} F_i,$$

$$\varphi_\alpha(H_i) = H_i, \quad \varphi_\alpha(E_i) = e^{d_i h \alpha_i} E_i, \quad \varphi_\alpha(F_i) = e^{-d_i h \alpha_i} F_i.$$

Furthermore, $U_h^G(\mathfrak{g})$ is quasitriangular by Theorem J (the conditions of this theorem are satisfied by $A = U_h(\mathfrak{b}_+)$ and $B = \widetilde{U}_h(\mathfrak{b}_-)$). For example, for $\mathfrak{g} = sl_2$ and $G = \mathbb{C}[[h]]$, the R-matrix of $U_h^G(sl_2)$ is given by

$$R_{\alpha,\beta} = e^{h(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(h)\, E^n \otimes F^n \in U_h^{\alpha}(sl_2) \mathbin{\widehat{\otimes}} U_h^{\beta}(sl_2)$$

for all $\alpha, \beta \in \mathbb{C}[[h]]$, where $R_n(h) = q^{n(n+1)/2} \frac{(1-q^{-2})^n}{[n]_q!}$ and $q = e^h$.

Appendix 7
Invariants of 3-dimensional G-manifolds from Hopf coalgebras
by Alexis Virelizier

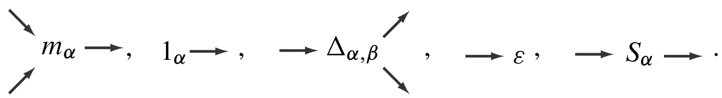
In this appendix we construct invariants of closed oriented 3-dimensional G-manifolds using Hopf G-coalgebras. In contrast to the methods of Chapters VII and VIII, we do not involve representations of the Hopf G-coalgebras. Our invariants generalize the Kuperberg invariant and the Hennings invariant of 3-manifolds corresponding to the case $G = \{1\}$.

Throughout this appendix, G is a group and K is a field.

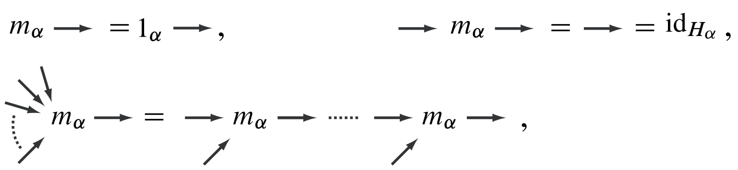
7.1 Kuperberg-type invariants

Kuperberg [Ku] derived an invariant of closed oriented 3-manifolds from any finite-dimensional involutory Hopf algebra. As the main geometric tool, he used Heegaard diagrams of 3-manifolds. Here we generalize Kuperberg's method to construct invariants of closed oriented 3-dimensional G-manifolds from involutory Hopf G-coalgebras.

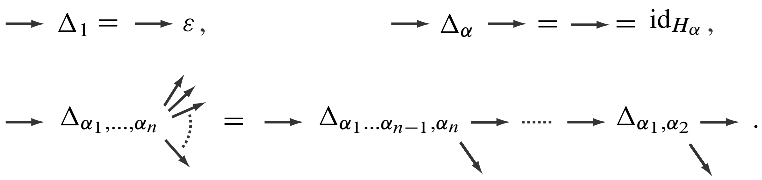
1.1 Diagrammatic formalism for Hopf G-coalgebras. Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf G-coalgebra of finite type. The multiplication $m_\alpha\colon H_\alpha \otimes H_\alpha \to H_\alpha$, the unit element $1_\alpha \in H_\alpha$, the comultiplication $\Delta_{\alpha,\beta}\colon H_{\alpha\beta} \to H_\alpha \otimes H_\beta$, the counit $\varepsilon\colon H_1 \to K$, and the antipode $S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}$ are represented pictorially as follows:



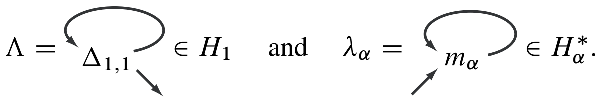
The inputs (incoming arrows) for multiplication are always ordered counterclockwise and the outputs (outgoing arrows) for comultiplication are always ordered clockwise. Furthermore, we adopt the following abbreviations:



and



The homomorphisms represented by such diagrams may be explicitly computed in terms of the structure constants. In particular, set



This means that if $(e_i)_i$ is a basis of H_1 and $C_i^{j,k} \in K$ are the structure constants of $\Delta_{1,1}\colon H_1 \to H_1 \otimes H_1$ defined by $\Delta_{1,1}(e_i) = \sum_{j,k} C_i^{j,k} e_j \otimes e_k$, then $\Lambda = \sum_{i,k} C_i^{i,k} e_k$. Likewise, if $(f_i)_i$ is a basis of H_α and $\mu_{i,j}^k \in K$ are the structure constants of m_α defined by $m_\alpha(f_i \otimes f_j) = \sum_k \mu_{i,j}^k f_k$, then $\lambda_\alpha(f_j) = \sum_k \mu_{k,j}^k$.

Assume now that H is involutory as defined in Appendix 6, Section 1.9. By Lemma 4 of [Vir4], $\lambda = (\lambda_\alpha)_{\alpha \in G}$ is a two-sided G-integral for H and Λ is a two-sided integral for H_1 such that

$$\lambda_1(1_1) = \varepsilon(\Lambda) = \lambda_1(\Lambda) = \dim H_1, \quad S_1(\Lambda) = \Lambda, \quad \text{and} \quad \lambda_{\alpha^{-1}} S_\alpha = \lambda_\alpha$$

for all $\alpha \in G$. By Lemma 5 of [Vir4], Λ and λ are symmetric in the following sense: for all $\alpha \in G$ and $x, y \in H_\alpha$,

$$\Delta_{\alpha,\alpha^{-1}}(\Lambda) = \sigma_{H_{\alpha^{-1}},H_\alpha} \Delta_{\alpha^{-1},\alpha}(\Lambda) \quad \text{and} \quad \lambda_\alpha(xy) = \lambda_\alpha(yx).$$

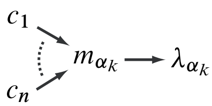
1.2 Construction of the invariant. Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be an involutory Hopf G-coalgebra of finite type such that the characteristic of the ground field K of H does not divide $\dim H_1$. Note that H is then semisimple and cosemisimple (see Appendix 6, Section 1.9).

Let (W, g) be a closed connected oriented 3-dimensional G-manifold. Recall from Section VII.2.1 that W is a closed connected oriented 3-dimensional manifold and g is a free homotopy class of maps from W to $X = K(G, 1)$. We present W by a Heegaard diagram (Σ, u, l), where Σ is an oriented closed surface of genus $g \geq 0$ embedded in W (and cutting W into two genus g handle bodies), $u = \{u_1, \ldots, u_g\}$ and $l = \{l_1, \ldots, l_g\}$ are two transverse g-tuples of pairwise disjoint circles embedded in Σ such that $\Sigma \setminus \bigcup_k u_k$ and $\Sigma \setminus \bigcup_i l_i$ are connected. We pick $z \in \Sigma \setminus (u \cup l)$ and orient all the circles u_k and l_i in an arbitrary way.

Traveling along each lower circle l_i, we obtain a word $w_i(x_1, \ldots, x_g)$ in the alphabet $\{x_1^{\pm 1}, \ldots, x_g^{\pm 1}\}$ as follows. Start at any point of l_i not belonging to u and make a round trip along l_i following its orientation. Begin with the empty word and each time l_i intersects some u_k, add on the right of the word the letter x_k if l_i intersects u_k positively[1] and the letter x_k^{-1} otherwise. After a complete turn along l_i, we obtain the word w_i. This word is defined up to conjugation due to the indeterminacy in the choice of the starting point on l_i. Since $\Sigma \setminus u$ is a 2-sphere with $2g$ disks deleted, there exists g loops $\gamma_1, \ldots, \gamma_g$ on (Σ, z) such that each γ_i intersects once positively u_i and does not meet $\bigcup_{j \neq i} u_j$. The homotopy classes $a_i = [\gamma_i] \in \pi_1(W, z)$, $i = 1, \ldots, g$ do not depend on the choice of the loops γ_i. By the Van Kampen theorem,

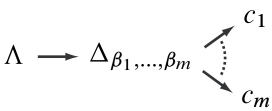
$$\pi_1(W, z) = \langle a_1, \ldots, a_g \mid w_i(a_1, \ldots, a_g) = 1 \text{ for } 1 \leq i \leq g \rangle.$$

For any $1 \leq k \leq g$, we provide the circle u_k with the label $\alpha_k = g_*(a_k) \in G$, where $g_* \colon \pi_1(W, z) \to \pi_1(X, x) = G$ is the homomorphism induced by a map $W \to X$ in the given homotopy class g carrying z to the base point x of $X = K(G, 1)$. To each u_k, we associate the tensor



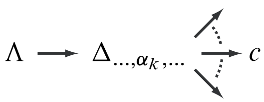
where $c_1, \ldots, c_n$ are the crossings between u_k and the circles l_i which appear in this order when making a round trip along u_k following its orientation. Since this tensor is cyclically symmetric (see Section 1.1), this assignment does not depend on the choice of the starting point on u_k.

To each circle l_i, we associate the tensor

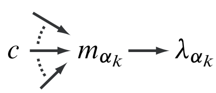


where $c_1, \ldots, c_m$ are the crossings between l_i and the circles u_k which appear in this order when making a round trip along l_i following its orientation; if l_i intersects u_k at c_j, then $\beta_j = \alpha_k \in G$ if the intersection is positive and $\beta_j = \alpha_k^{-1}$ otherwise. Note that $\beta_1 \ldots \beta_m = w_i(\alpha_1, \ldots, \alpha_g) = 1$ and so the tensor associated to l_i is well defined. Since this tensor is cyclically symmetric, this assignment does not depend on the choice of the starting point on l_i.

Let c be a crossing point between some u_k and l_i. If l_i intersects u_k positively at c, then we contract the tensors

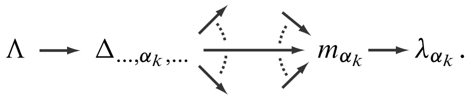


and

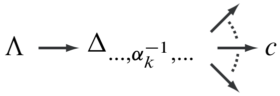


associated to l_i and u_k as follows:

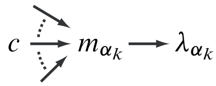
[1] An oriented curve γ on Σ intersects *positively* another oriented curve ρ on Σ at a point $c \in \Sigma$ if $(d_c\gamma, d_c\rho)$ is a positively-oriented basis for $T_c\Sigma$.



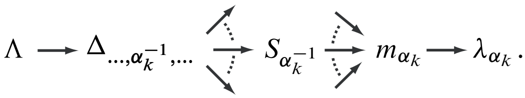
If l_i intersects u_k negatively at c, then we contract the associated tensors



and



as follows:



After having done such contractions at all the crossing points, we obtain an element $Z(\Sigma, u, l)$ of K. Set

$$\mathrm{Ku}_H(W, g) = (\dim H_1)^{-g}\, Z(\Sigma, u, l) \in K.$$

Theorem A ([Vir4], Theorem 9). $\mathrm{Ku}_H(W, g)$ *is a homeomorphism invariant of the G-manifold* (W, g).

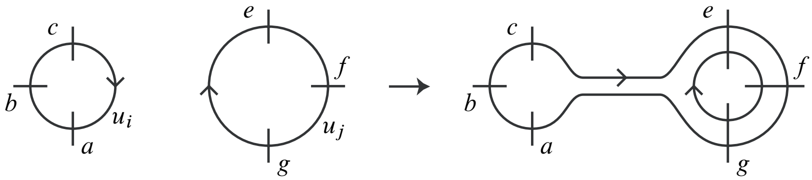
1.3 Examples. 1. If g is the trivial homotopy class of maps $W \to X$ represented by the map $W \to \{x\} \subset X$, then $\mathrm{Ku}_H(W, g)$ is equal to the Kuperberg invariant of W derived from the involutory Hopf algebra H_1. In particular, for $G = \{1\}$, we recover the Kuperberg invariant.

2. Let G, L be finite groups and $\mathbb{C}^G, \mathbb{C}^L$ be the Hopf algebras of $\mathbb{C}$-valued functions on G and L, respectively. A group homomorphism $\phi\colon L \to G$ induces a Hopf algebra morphism $\mathbb{C}^G \to \mathbb{C}^L$, $f \mapsto f\phi$ whose image is central. By Appendix 6, Section 1.2, this data yields to a Hopf G-coalgebra $H^\phi = \{H^\phi_\alpha\}_{\alpha\in\pi}$, which is involutory and of finite type. Note that $H^\phi_\alpha \cong \mathbb{C}^{\phi^{-1}(\alpha)}$ as an algebra. For every closed connected oriented 3-dimensional G-manifold (W, g),

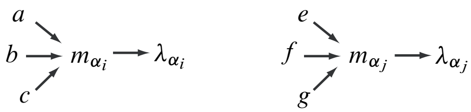
$$\mathrm{Ku}_{H^\phi}(W, g) = \#\{f : \pi_1(W, z) \to L \mid \phi f = g_*\},$$

where $z \in W$ and $g_*\colon \pi_1(W, z) \to G$ is the homomorphism induced by a map $W \to X$ in the homotopy class of g carrying z to the base point x of $X = K(G, 1)$.

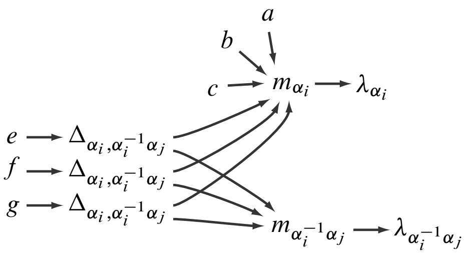
1.4 Proof of Theorem A (sketch). The proof is based on a "G-colored" version of the Reidemeister-Singer theorem which relates any G-colored Heegaard diagrams representing (W, g, z). For example, suppose that a circle u_i (with label α_i) slides across another circle u_j (with label α_j). Assume, as a representative case, that both circles have three crossings with $\bigcup_k l_k$:



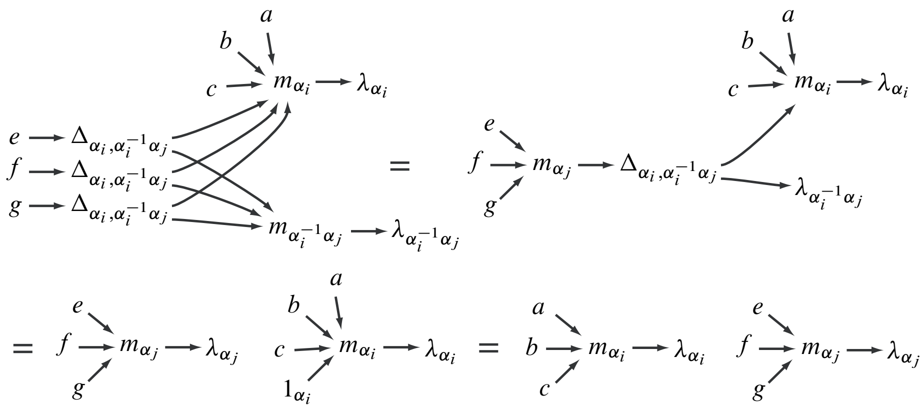
Using the anti-multiplicativity of the antipode (which allows us to consider only the positively-oriented case of the contraction rule), we obtain that the factor



of $Z(\Sigma, u, l)$ is replaced under this move by



Since the comultiplication is multiplicative and $\lambda = (\lambda_\alpha)_{\alpha \in G}$ is a left G-integral for H, these two factors are equal:



For a detailed proof of Theorem A, we refer to [Vir4] (cf. also the next remark).

1.5 Remark. Since H is of finite type, semisimple, and spherical (with spherical elements $w_\alpha = 1_\alpha \in H_\alpha$), the category $\mathrm{Rep}(H)$ of finite-dimensional representations of H is a finite semisimple spherical G-category. Hence we can consider the state sum invariant $|W, g|_{\mathrm{Rep}(H)}$ introduced in Appendix 2. Adapting the arguments of [BW1], we obtain that

$$\mathrm{Ku}_H(W, g) = |W, g|_{\mathrm{Rep}(H)}$$

for any closed connected oriented 3-dimensional G-manifold (W, g).

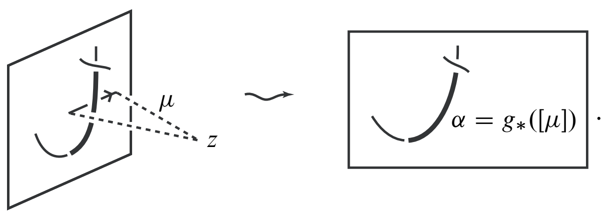
7.2 Hennings–Kauffman–Radford-type invariants

Hennings [He] derived invariants of links and 3-manifolds from right integrals on certain Hopf algebras. His construction was studied and clarified by Kauffman and Radford [KaRa]. In this section, we generalize the Hennings–Kauffman–Radford method to construct an invariant τ_H of 3-dimensional G-manifolds by using a ribbon Hopf G-coalgebra H. When the ribbon G-category Rep(H) of representations of H is modular, we compare τ_H with the Turaev invariant $\tau_{\mathrm{Rep}(H)}$ from Section VII.2.

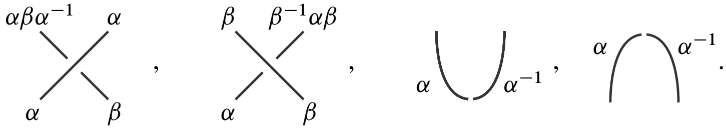
2.1 The invariant τ_H. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a unimodular ribbon Hopf G-coalgebra of finite type and let $\lambda = (\lambda_\alpha)_{\alpha \in G}$ be a (non-zero) right G-integral for H such that $\lambda_1(\theta_1^{\pm 1}) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha \in G}$ is the twist of H.

Let (W, g) be a closed connected oriented 3-dimensional G-manifold (see Section VII.2.1). We define $\tau_H(W, g) \in K$ as follows. Present W as the result of surgery on S^3 along a framed link ℓ with $m = \#\ell$ components. Recall that W is obtained by gluing m solid tori to the exterior E_ℓ of $\ell \in S^3$. Take any point $z \in E_\ell \subset W$. Pick in the homotopy class g a map $W \to X$ carrying z to the base point x of X. The restriction of this map to E_ℓ induces a homomorphism $g_* \colon \pi_1(E_\ell, z) \to \pi_1(X, x) = G$. Note that the triple (ℓ, z, g_*) is an unoriented special G-link in the sense of Section VI.5.4.

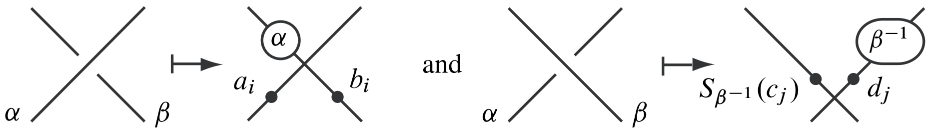
Regularly project ℓ onto a plane from the base point z, that is, consider a diagram of ℓ such that the base point z corresponds to the eyes of the reader. Without loss of generality, we can assume that the extremal points of the diagram with respect to a chosen height function are isolated. Label the vertical segments of the diagram (delimited by the extremal points of the height function and the under-crossings) by elements of G in the following way: a vertical segment is labeled by $\alpha = g_*([\mu]) \in G$ where μ is a meridional loop of ℓ based at z which encircles the segment once so that its linking number with this segment oriented downwards is +1:



At the crossings and the extremal points the labels are related as follows:



Now, we decorate each crossing of the diagram of ℓ with the R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta\in G}$ of H and with small disks labeled by elements of G as follows:

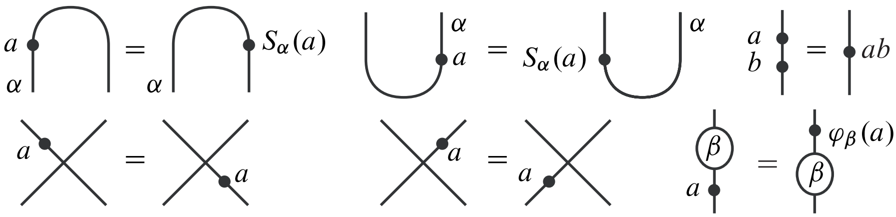


where

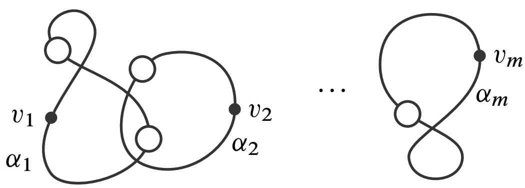
$$R_{\alpha,\beta} = a_i \otimes b_i \quad \text{and} \quad R_{\beta^{-1},\alpha} = c_j \otimes d_j.$$

In this formalism it is understood that there is a summation over all the pairs a_i, b_i and c_j, d_j. The diagram obtained at this step is composed by $m = \#\ell$ transverse closed plane curves (possibly endowed with G-labeled disks), each of them arising from a component of ℓ.

We use the following rules to concentrate the algebraic decoration of each of these plane curves in one point (distinct from the extremal points and the labeled disks):

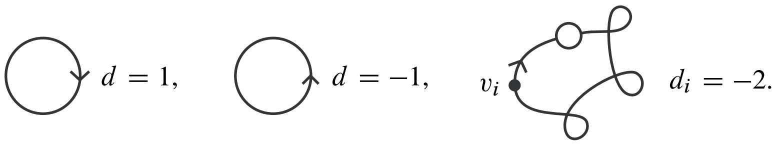


This gives m elements $v_1 \in H_{\alpha_1}, \ldots, v_m \in H_{\alpha_m}$:



If there is no algebraic decoration on the i-th curve then, by convention, $v_i = 1_{\alpha_i}$.

For $1 \leq i \leq m$, let d_i be the Whitney degree of the i-th curve obtained by traversing it upwards from the point where the algebraic decoration has been concentrated. The Whitney degree is the algebraic number of turns of the tangent vector of the curve. For example:



Finally set

$$\tau_H(W,g) = \lambda_1(\theta_1)^{b_-(\ell)-m}\ \lambda_1(\theta_1^{-1})^{-b_-(\ell)}\ \lambda_{\alpha_1}(w_{\alpha_1}^{1+d_1} v_1)\ldots\lambda_{\alpha_m}(w_{\alpha_m}^{1+d_m} v_m) \in K,$$

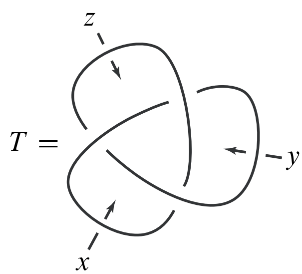
where $w = (w_\alpha)_{\alpha\in G}$ is the spherical G-grouplike element of H (see Appendix 6, Section 2.6) and $b_-(\ell)$ is the number of strictly negative eigenvalues of the linking matrix of ℓ (with the framing numbers on the diagonal).

Theorem B ([Vir1], Theorem 4.12). *$\tau_H(W,g)$ is a homeomorphism invariant of the G-manifold (W,g).*

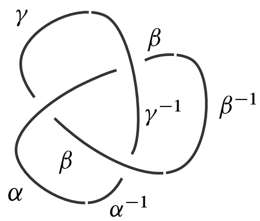
The invariant τ_H is preserved under multiplication of the right G-integral λ by any element of K^*. Since the space of right G-integrals for H is one-dimensional (see Appendix 6, Section 1.1), τ_H does not depend on the choice of λ.

2.2 Examples. 1. If g is the trivial homotopy class of maps $W \to X$ represented by the map $W \to \{x\} \subset X$, then $\tau_H(W,g)$ is the Hennings invariant (in its Kauffman–Radford reformulation) of W derived from the ribbon Hopf algebra H_1. In particular, when $G = \{1\}$, we recover the Hennings invariant.

2. Let P be the closed oriented 3-dimensional manifold obtained by surgery along the trefoil T with framing $+3$:



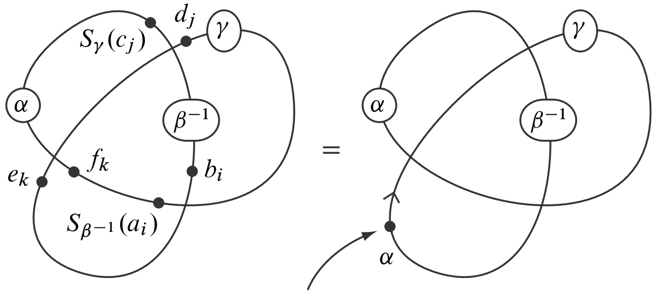
The Wirtinger presentation of the group of T is $\langle x,y,z \,|\, xy = yz = zx\rangle$ and $\pi_1(P) = \langle x,y,z \,|\, xy = yz = zx,\ \ xzy = 1\rangle$. Let g be a free homotopy class of maps $P \to X = K(G,1)$ inducing a homomorphism $g_*\colon \pi_1(T) \to G$. Set $\alpha = g_*(x)$, $\beta = g_*(y)$, $\gamma = g_*(z)$. The labeling of the vertical segments of the diagram of T is:



Expand

$$R_{\beta^{-1},\alpha^{-1}} = \sum_i a_i \otimes b_i, \quad R_{\gamma,\alpha} = \sum_j c_j \otimes d_j, \quad \text{and} \quad R_{\alpha,\beta} = \sum_k e_k \otimes f_k.$$

The algorithm above gives



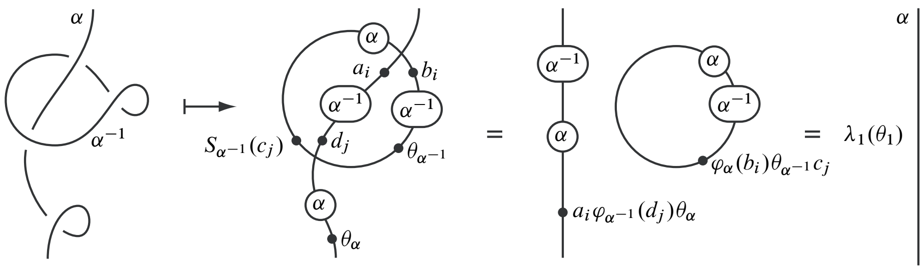
$$\varphi_{\gamma^{-1}} S_\beta^{-1}(a_i S_{\beta^{-1}}^{-1}(f_k)) d_j e_k S_{\alpha^{-1}}(b_i) S_{\alpha^{-1}} \varphi_\beta S_\gamma(c_j)$$

Therefore

$$\tau_H(W,g) = \lambda_1(\theta_1)^{-1} \sum_{i,j,k} \lambda_\alpha\big(w_\alpha^3 \varphi_{\gamma^{-1}} S_\beta^{-1}(a_i S_{\beta^{-1}}^{-1}(f_k)) d_j e_k S_{\alpha^{-1}}(b_i) S_{\alpha^{-1}} \varphi_\beta S_\gamma(c_j)\big).$$

Exercise. Given a finite group G, compute $\tau_{D_G(G)}(W,g)$, where $D_G(G)$ is the ribbon Hopf G-coalgebra defined in Appendix 6, Section 3.3.

2.3 Proof of Theorem B (sketch). The proof uses a "G-colored" version of the Kirby calculus relating G-colored link diagrams representing (W,g). For example, the invariance under the G-colored Fenn–Rourke move with one strand is proven as follows.



where

$$R_{\alpha,\alpha^{-1}} = \sum_i a_i \otimes b_i \quad \text{and} \quad R_{\alpha^{-1},\alpha} = \textstyle\sum_j c_j \otimes d_j .$$

Indeed,

$$\begin{aligned}
&\sum_{i,j} \lambda_{\alpha^{-1}}(\varphi_\alpha(b_i)\theta_{\alpha^{-1}} c_j)\, a_i \varphi_{\alpha^{-1}}(d_j)\theta_\alpha \\
&\quad = \sum_{i,j} \lambda_{\alpha^{-1}}(\theta_{\alpha^{-1}} b_i c_j)\, \theta_\alpha \varphi_\alpha(a_i) d_j \\
&\quad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})((\theta_{\alpha^{-1}} \otimes \theta_\alpha)(\sigma_{\alpha,\alpha^{-1}}(\varphi_\alpha \otimes \mathrm{id}_{H_{\alpha^{-1}}})(R_{\alpha,\alpha^{-1}}))R_{\alpha^{-1},\alpha}) \\
&\quad = (\lambda_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(\theta_1) = \lambda_1(\theta_1)\, 1_\alpha .
\end{aligned}$$

These equalities follows from the properties of the crossing φ, the twist θ, and the G-integral λ of H (see Appendix 6). We refer to [Vir1] for a detailed proof. A crucial role in the proof is played by the fact that the family of homomorphisms $(H_\alpha \to K,\ x \mapsto \lambda_\alpha(w_\alpha x))_{\alpha \in G}$ is a G-trace for H (Appendix 6, Theorem G).

2.4 Comparison with the Turaev invariant. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a ribbon Hopf G-coalgebra of finite type. Suppose that the ribbon G-category $\mathrm{Rep}(H)$ of representations of H (see Section VIII.1.7) is modular. Then the Turaev invariant $\tau_{\mathrm{Rep}(H)}$ of Section VII.2 is well defined. Moreover, under these assumptions, H is unimodular, since H_1 is then factorizable and so unimodular (see [Sw]). Furthermore, $\lambda_1(\theta_1^{\pm 1}) \neq 0$ for every non-zero right G-integral $\lambda = (\lambda_\alpha)_{\alpha \in G}$ for H (since $\lambda_1(\theta_1^{\pm 1}) = \Delta_\pm^{\mathrm{Rep}(H)}$ up to a non-zero scalar multiple). Hence the invariant of the preceding section τ_H is also defined. The next theorem shows that under these assumptions, the invariants $\tau_{\mathrm{Rep}(H)}$ and τ_H are essentially equivalent.

Theorem C ([Vir1], Theorem 4.18). *Let H be a ribbon Hopf G-coalgebra of finite type such that its ribbon G-category of representations* $\mathrm{Rep}(H)$ *is modular. Then the invariant τ_H is well defined and for every closed connected oriented 3-dimensional G-manifold (W, g),*

$$\tau_{\mathrm{Rep}(H)}(W, g) = D^{-1} \left(\frac{D}{\Delta_-} \right)^{b_1(W)} \tau_H(W, g),$$

where $b_1(W)$ is the first Betti number of W, and D, Δ_- are as in Section VII.1.7.

The proof is based on a description of the Turaev invariant in terms of the coend of the G-category $\mathrm{Rep}(H)$, see [Vir1].

Note that when the category $\mathrm{Rep}(H)$ is not modular (typically, when H is not semisimple, see Appendix 6, Section 1.7) the invariant $\tau_{\mathrm{Rep}(H)}$ is not defined while τ_H may be defined.

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